A FIRST COURSE
IN
NOMOGRAPHY

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PREFACE

In many branches of science, in engineering practice, in technology, in industry and in military science, Nomography is a recognised means of carrying out graphical calculations. The ballistic constant in gunnery, flame temperature in the research of coal-gas combustion, the angle of twist in a thread of given thickness with a given number of turns per inch, the conversion of counts in the textile industry, can all be calculated by means of nomograms. Nomographic charts are simple and certain in use, so that calculations formerly entrusted to skilled and responsible computers can now be safely left to the care of a comparatively unskilled subordinate. It is the object of this First Course to offer a clear and elementary account of the construction and use of such charts.

The method of treatment chosen is based on experience gained in the making of nomograms for various technological departments in the University of Leeds, and in other ways. It is a treatment that should be found useful by the reader who desires to become acquainted both with the theory of nomography and with its practical use. Chapter III. begins the nomography proper, but the reader is advised to study Chapters I. and II. first, in order to see how the nomograms in Chapter III. can be constructed. Special attention is directed to §§ 49-50 in Ch. IV., and to Chapter VIII. Answers have been purposely omitted, even where the examples lead to numerical results.

The sincere thanks of the author are due to Prof. W. P. Milne, M.A., D.Sc., Mr. R. C. Fawdry, M.A., and Mr. A. W. Siddons, M.A., for their kind help in reading the manuscript and making many suggestions, most of which have been adopted. Thanks are also due to Mr. R. M. Milne, M.A., for suggesting an important example in gunnery, and to the Editor of the *Mathematical Gazette* for permission to make use of the subject matter and diagrams of an article by the author published in that journal. Extensive use has been made of d'Ocagne's *Traité de Nomographie*, and the author's great indebtedness to this admirable work is gladly acknowledged.

Special thanks are due to Mr. A. J. V. Umanski, M.Sc., for his kindness in reading the proofs and detecting several errors. The author will be very grateful if users of the book will inform him of any mistakes they may find, or of any suggestions they can offer for increasing the usefulness of the book.

LEEDS, *January*, 1920. S. B.

PREFACE TO SECOND EDITION

In this edition a chapter has been added on the use of determinants in the construction of nomograms. Several corrections and small alterations have been made, mainly at the instance of a number of kind friends, who have taken the trouble to suggest them to me, and whom I thank most sincerely. S. B.

LEEDS, *March*, 1925.

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LIST OF DIAGRAMS AND INDEX OF NOMOGRAMS

HISTORICAL SKETCH

WHEN Descartes invented Coordinate Geometry, he put at the disposal of mathematicians a powerful weapon that has led to phenomenal advances in all branches of mathematical science. For the purpose of practical use in calculations by means of graphical representation on a plane, the number of variables that can be used is obviously restricted to two. This limitation was removed by Buache (1752), who introduced the method of contours—now incorporated in all atlases and surveys. By means of contours it is possible to deal with three quantities at once, whilst they are all represented on one plane, as *e.g.* in indicating the variation in the height of land or the depth of the sea. This sufficed for a time, but the extraordinary growth of railway systems all over the world led to important developments by Lalanne (1841), Massau (1884), and Lallemand (1886).

The idea of using collinear points, which constitutes the chief beauty of the method of the present book, was developed by d'Ocagne (1884). It was d'Ocagne, too, who applied the name NOMOGRAPHY to this method, in his book *Les calculs usuels effectués au* *moyen des abaques* (1891). Since then further extensions have been made by d'Ocagne and others.

In recent years the utility and convenience of nomography have been increasingly realised, and the subject has gained in importance and recognition, particularly in engineering practice. It is, in the main, a product of French mathematical genius. Articles have appeared in one or two English journals, and excellent accounts of the subject in English are to be found in Hezlet's *Nomography* (Royal Military Institution, Woolwich) and Lipka's *Graphical and Mechanical Computation* (Wiley, New York). Other books which may be consulted are Rose's *Line Charts for Engineers* (Chapman & Hall, London), and Hewes and Seward's *Design of Diagrams for Engineering Formulas* (McGraw Hill, London).

But the reader who is interested in the subject cannot do better than read d'Ocagne's excellent *Traité de Nomographie* (G. Villars, Paris, 1899), as well as *Principes Usuels de Nomographie* by the same author and publishers.

NOMOGRAPHY

INTRODUCTION

1. Object of Nomography.

We are all familiar with graph-drawing as a means of solving equations. But ordinary graphical methods are often inconvenient, because a separate graph is required for practically each equation we have to solve. Thus, to solve the quadratic equation

$$x^2+x-2=0,$$

we need to draw the graph $y=x^2+x$, and find where it is cut by the line $y=2$. To solve the equation $x^2+2x-2=0$, we must draw the graph $y=x^2+2x$, and similarly for other values of the coefficient of x. *The object of Nomography is to enable us to solve* all *equations of a given type by means of* one *diagram.* Thus all quadratic equations of the type $x^2+ax+b=0$ can be solved by means of one graph which can be drawn once for all. In the same way all cubic equations of, say, the form $x^3+ax+b=0$ can be solved graphically by means of one diagram. Such a diagram is called a **Nomogram.**

It is also the object of Nomography to enable us to find the value of a complicated expression graphically. Let us consider, *e.g.*, the formula for the pressure R in pounds on a flat plate normal to a passing current of air, viz.,

$$R = 0{\cdot}0194 WSV^2 \text{ (lbs.)},$$

where W is the weight of the air in pounds per cubic foot, S is the area of the plate in square feet, and V is the relative velocity of the air in feet per second. If the pressure has to be found for a number of different plates at various velocities, it is obviously convenient to have a diagram which, having been constructed once for all, can be used for any values of W, S, and V that are likely to occur.

In the present volume we shall consider in a simple manner how to construct and use nomograms for multiplication and division in formulae like the one for air pressure, as well as the more general methods applicable to easy algebraic and other equations. As an introduction to the nomograms for multiplication and division we shall consider first the construction and use of nomograms for addition and subtraction.

2. Method of Nomography.

The method of Nomography is as follows. Suppose that a number x is determined when two numbers a, b are given; *e.g.* let $x = ab$ or a/b or $3a^2/b^3$, or let x be given by

$$x^2 + ax + b = 0,$$

or

$$a \sin x + b \cos x + 1 = 0\ ;$$

then two straight lines a, b are graduated (Fig. 1), and a curve (which we shall, when necessary, call the x curve) is also graduated in such a way that, if a straight line is drawn joining the graduation a on scale a to the graduation b on scale b, the line cuts the x curve at the graduation x, where x is the desired

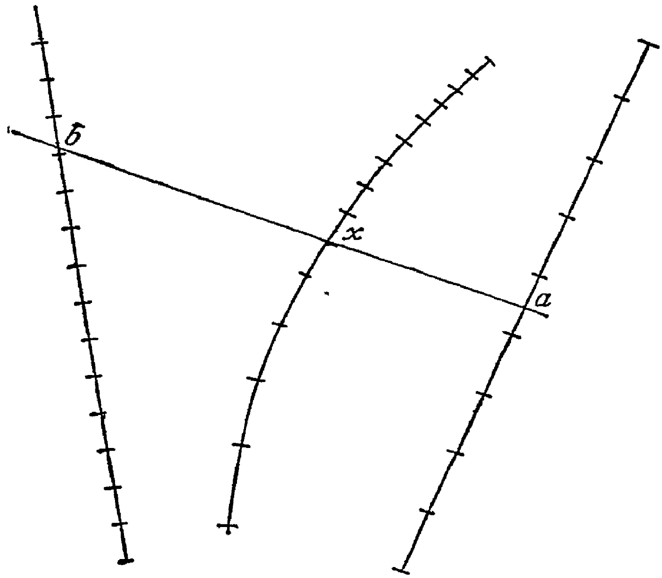


FIG. 1.

result. Thus, in Fig. 2, the line joining the point a on the scale a to the point b on the scale b cuts the curve x at a point whose x coordinate (or abscissa) is a solution of the equation $x^2+ax+b=0$.

Sometimes the x curve and the a, b scales are three parallel lines. In Fig. 3 we have a nomogram for the formula

$$R=0{\cdot}0194WSV^2.$$

already mentioned. If the line joining the appropriate point on the W scale to the appropriate point on the S scale is made to cut the "reference line," and this point of intersection is joined to the appropriate point on the V scale, the join cuts the R scale at the graduation giving the value of R. The theory

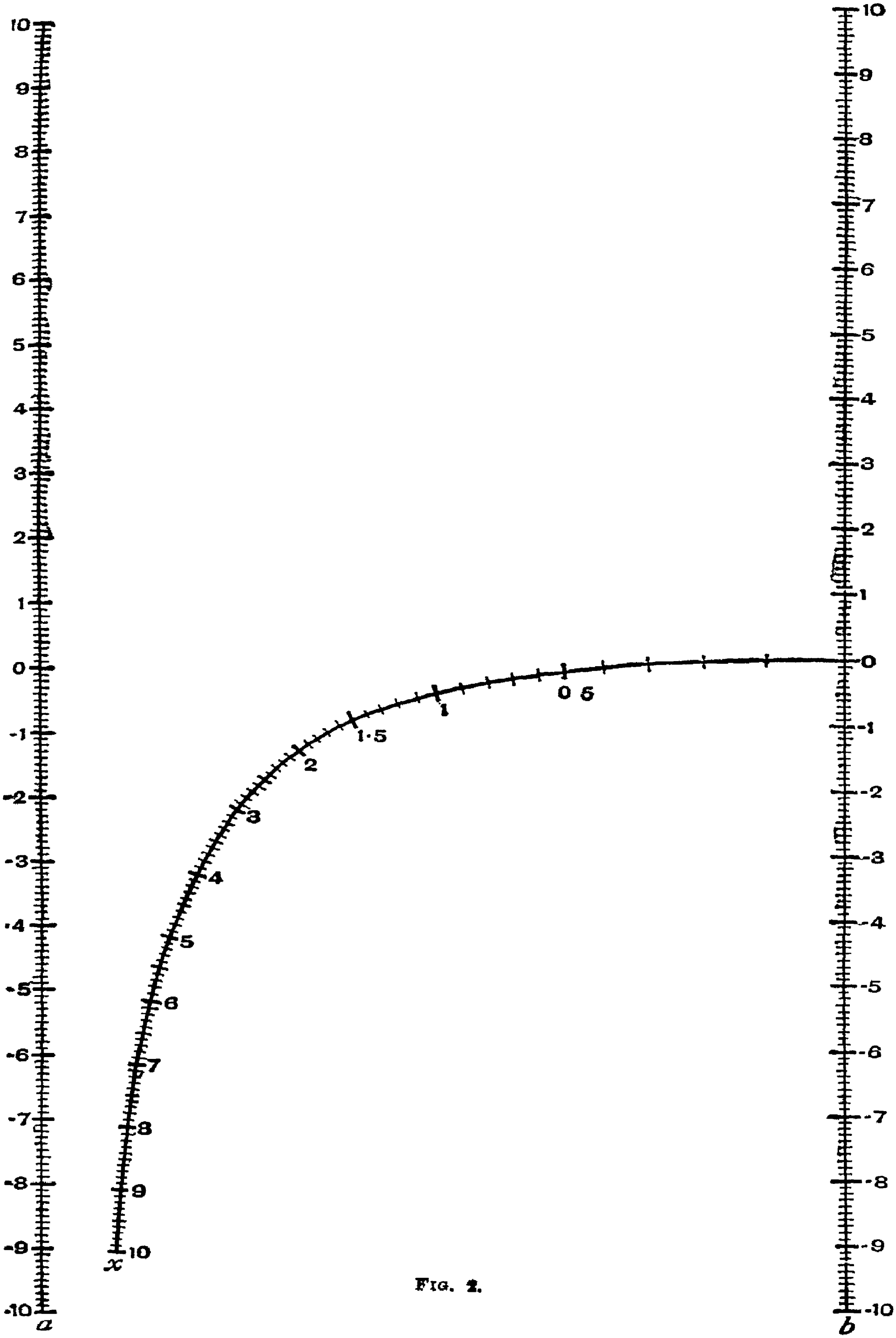


Fig. 2.

of such nomograms is very much easier than that of more general types. We shall, then, commence with

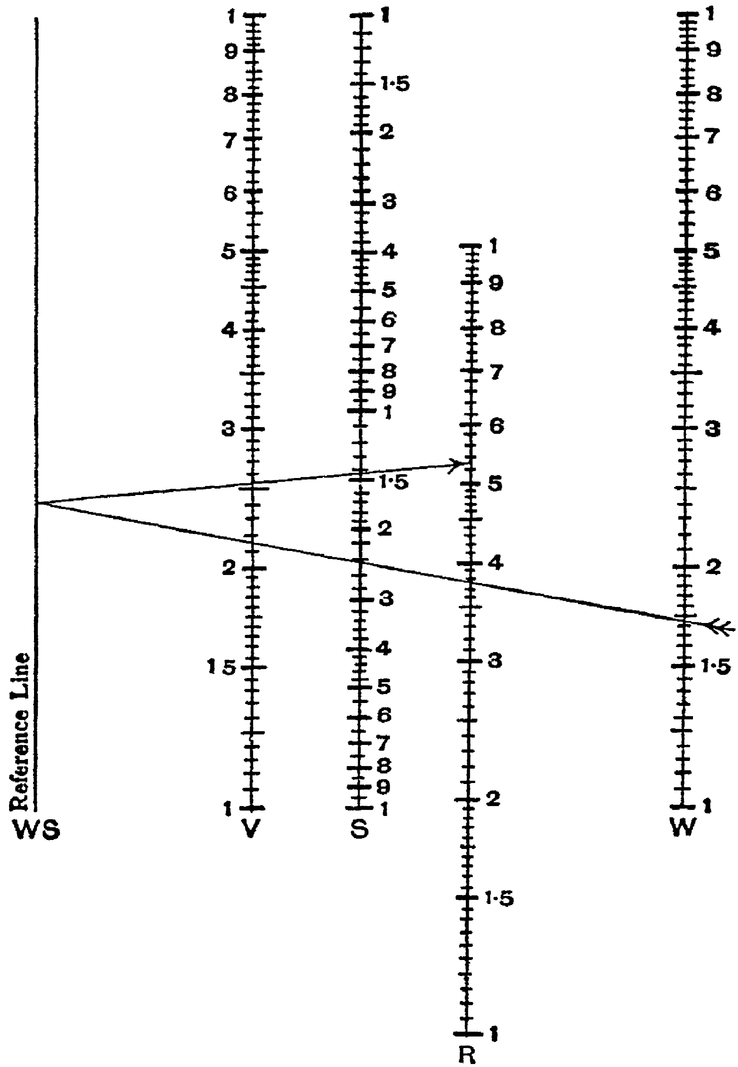


FIG. 3.

these simple forms, which include all cases involving addition and subtraction, or multiplication and division of numbers raised to given powers.

CHAPTER I

NOMOGRAMS FOR ADDITION AND SIMULTANEOUS EQUATIONS

3. Nomogram for Addition

In Fig. 4 we have three parallel straight lines: the outer ones, a, b, are graduated with the same unit, the middle one, x, is midway between a and b, and is graduated with half their unit. The zeros are collinear; in our case they are all on a line perpendicular to the scales. If now we take the graduation a on scale a, and the graduation b on scale b, the line joining them cuts scale x in the graduation $a+b$. This is because

twice distance $x =$ *distance* $a +$ *distance* b,

and the unit used to measure the distance Ox is purposely made half of that used for Oa and Ob, so that

graduation $x =$ *graduation* $a +$ *graduation* b.

Thus we have in Fig. 1 a nomogram for addition. If any straight line cuts the three scales at graduations a, b, x, we have $x=a+b$.

This is the principle of nomography The expression $a+b$ represents a *type of calculation*, namely the addition of two quantities. For this type or law (*nomos* =law in Greek) the scales in Fig. 4 can be used for all values of a and b. Theoretically there is no restriction on these quantities; but if we have large values of a and b we need an inconveniently long

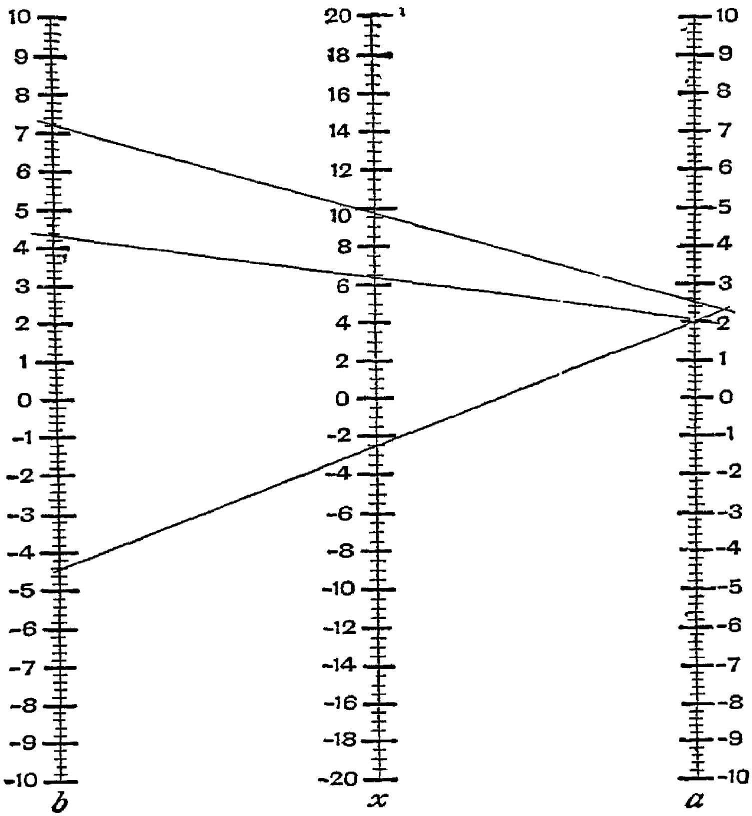


FIG 4.

figure. This is obviated by taking out a common factor in the form of a multiple of 10. Thus we graduate a and b from -10 to $+10$, and x from -20 to $+20$, with subdivisions as far as is possible—this, naturally, is determined by the size of the diagram. In Fig. 4, which is made small for convenience of printing, fifths are given for a and b, so that fiftieths can be estimated very approximately by the eye, and halves for x, so that twentieths can be estimated. Then to add up 255 and 720, we take the point 2·55 on scale a, and the point 7·20 on scale b. The line joining these points cuts the scale x at the graduation 9·75, so that the number required is 975.

Very small quantities can be treated in the same way; thus, to add up 0·0205 and 0·0435, we take the point 2·05 on scale a, and the point 4·35 on scale b. The line joining them cuts the scale x at the point 6·40, so that the sum required is 0·0640.

It is of course clear that we would not in practice want to use a nomogram for mere addition; but, as the fundamental ideas of nomography are well illustrated by the elementary problem of addition and subtraction, it is well that the reader should study the use of nomography for addition and subtraction, and so make himself thoroughly familiar with the idea and the underlying principles.

4. Extended Nomogram for Addition.

Suppose now we wish to find the values of $a+2b$ for all different values of a and b, *i.e.* we desire a nomogram for the type $x=a+2b$. We take scales a, b as before (Fig. 5), graduated with the same unit, but let the third scale x be twice as far from a as it is from b, and let its unit of graduation be one-third of the unit used in a, b. Let a straight line cut the scale a at the point a and the scale b at the point b.

If it cuts the scale x at the point x, then, by elementary geometry, we have that

three times distance x = *distance* a + *twice distance* b.

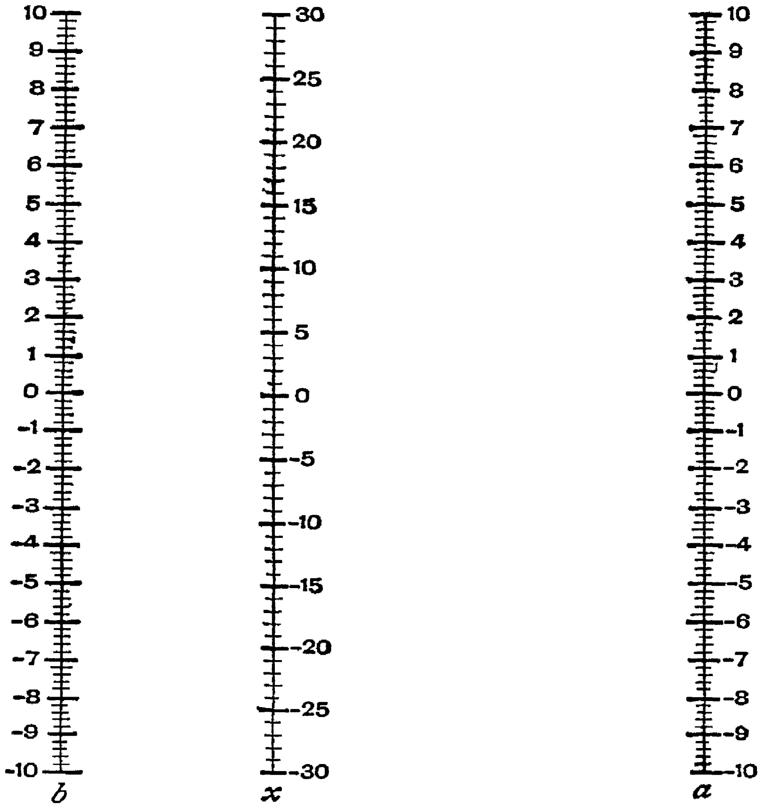


FIG. 5.

As the unit used in the scale x is one-third of the unit used in scales a, b, it follows that

graduation x = *graduation* a + *twice graduation* b.

5. Negative Quantities.

It is clear that the nomograms in Figs. 4 and 5 can be used for negative values of a, b as well as for positive values. Thus, in Fig. 4, if we take 2·00 on scale a and $-4{\cdot}45$ on scale b, we get the result $-2{\cdot}45$

on scale x. Although we would in ordinary *arithmetic* call this subtraction, yet, for purposes of nomography, we shall adopt the *algebraic* method of using the term addition even if we have negative quantities. Thus we shall talk about $a+b$ or $a+2b$ and use Figs. 4 and 5 for all values of a and b, with their proper signs attached. The sign of x will be given automatically by the nomogram. We shall consider expressions like $a-b$, $a-2b$, later.

6. Generalised Nomogram for Addition.

Using the idea of Fig. 5, we can now construct a nomogram for $x = la + mb$, where l, m are two *given*

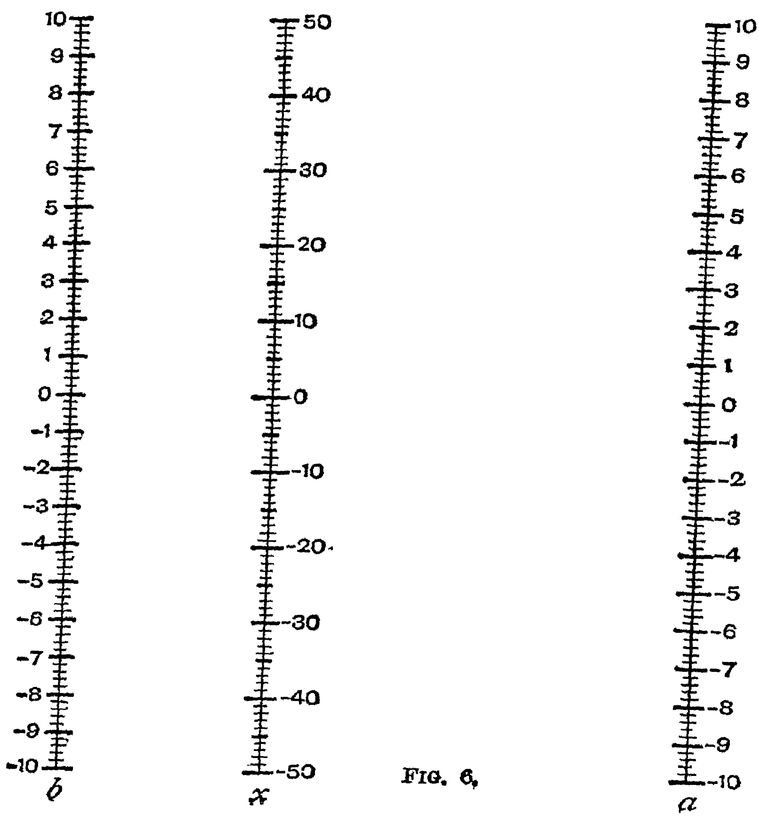


FIG. 6.

positive quantities. We take scales a, b as before, but place the scale x so that its distance from the scale a is to its distance from the scale b in the ratio $m : l$. If a straight line cuts off *distances* a on scale a, b on scale b, and x on scale x, then

$$(l+m) \times \textit{distance } x = l \times \textit{distance } a + m \times \textit{distance } b,$$

by easy geometry of similar triangles. Now let us graduate the scale x so that its unit is $\frac{1}{l+m}$ of the unit used in scales a, b. Then we have the result:

$$\textit{graduation } x = l \times \textit{graduation } a + m \times \textit{graduation } b,$$

so that $$x = la + mb.$$

In Fig. 4 we had $l=1$, $m=1$; in Fig. 5 we have $l=1$, $m=2$. In Fig. 6 we have used $l=1{\cdot}5$, $m=3{\cdot}5$, so that the distance of the scale x from the scale a is to its distance from scale b in the ratio $3{\cdot}5 : 1{\cdot}5$, *i.e.* $7 : 3$, and the unit used in scale x is $\frac{1}{1{\cdot}5+3{\cdot}5}$, *i.e.* $\frac{1}{5}$ of the a, b unit.

7. Simultaneous Simple Equations. Nomograms with Constant Unit

It is clear that the use of a unit $\frac{1}{l+m}$ in the x scale is not convenient, for two reasons. Firstly, $l+m$ may be a large quantity, so that the x unit may become too small. Secondly, we must in practice avoid the use of widely differing units as far as possible, as well as the necessity for regraduating too often. But, at present, in view of the intended application to simultaneous simple equations, we shall introduce only a slight modification, which we shall illustrate by reference to one or two simple cases.

In Fig. 7 the scales a, b, x, y are all graduated in exactly the same way, but the scale x is midway

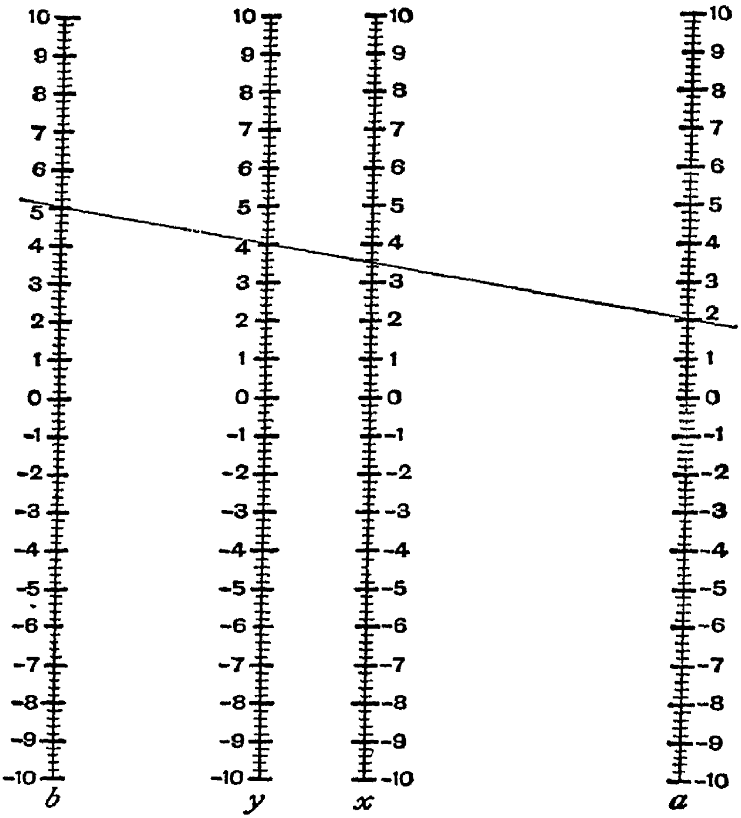


FIG. 7.

between scales a, b, and the scale y is twice as far from a as it is from b. Let a line cut all the four scales. It is easy to show that

twice distance x = *distance* a + *distance* b,

3 *times distance* y = *distance* a + *twice distance* b.

Hence we get, since all the scales are graduated alike,

twice graduation x = graduation a + graduation b,

3 *times graduation y = graduation a + twice graduation b.*

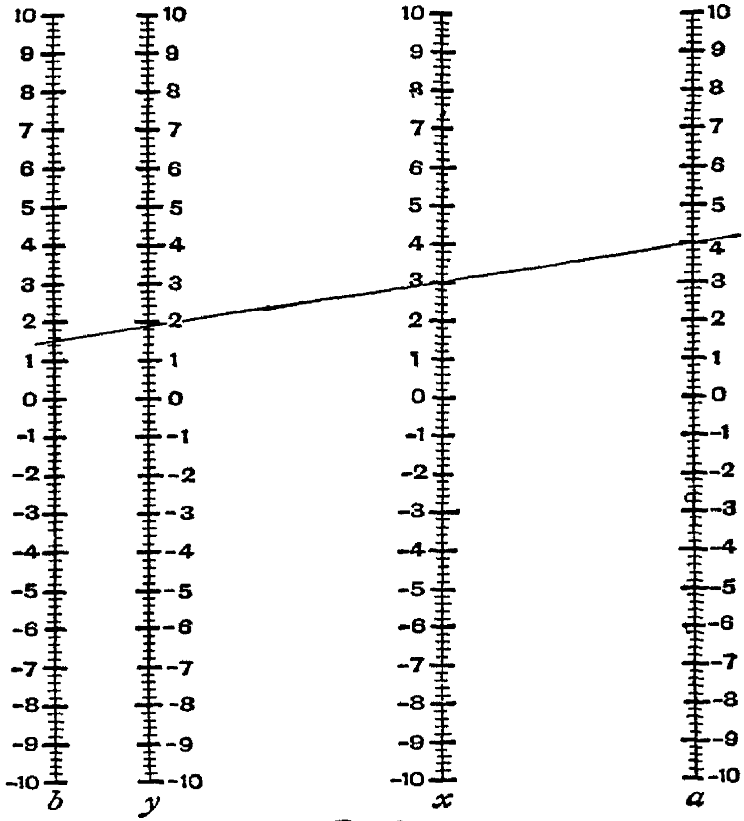


FIG 8.

If now we consider a and b as two unknown quantities, we have for them the two equations,

$$a + b = \textit{twice graduation } x,$$

$$a + 2b = 3 \textit{ times graduation } y.$$

Suppose now we have to solve the simultaneous equations $\quad a + b = 7, \quad a + 2b = 12$;

we take the graduation 7/2, *i.e.* 3·5 on scale x, and the graduation 12/3, *i.e.* 4 on scale y: the line joining these points cuts the scales a, b in the graduations $a=2$, $b=5$, which are the solutions required.

In the same way, if we have to solve the equations

$$3a+2b=15, \quad a+6b=13,$$

we write the equations in the form

3 *times graduation* a + *twice graduation* b
= 5 *times graduation* x,
graduation a + 6 *times graduation* b
= 7 *times graduation* y,

so that the graduation x to be used is 15/5, *i.e.* 3, and the graduation y to be used is 13/7, *i.e.* 1·86. In Fig. 8 the scales a, b, x, y are all graduated alike, but the x scale is 2 : 3 as far from a as it is from b, and the y scale is 6 times as far from a as from b. If we join the graduation 3 on the x scale to the graduation 1·86 on the y scale, the line cuts the a, b scales in the solutions required, namely

$$a=4, \quad b=1{\cdot}5.$$

We can now proceed to the general problem.

8. Simultaneous Equations.

Let the scales a, b, x be all graduated with the same unit, so that we have three exactly equal scales, the distance of the scale x from the scales a, b being in the ratio $m : l$. A line which cuts the scales at graduations a, b, x will give us the value of $la+mb$, if we multiply the graduation x by $l+m$.

Take now any point x (*i.e.* graduation $\frac{x}{l+m}$ on the scale x): any line through it will cut the a, b scales at graduations which are such that $la+mb=x$.

There are an infinite number of such lines through the point x, and, therefore, there are an infinite number of pairs of values a. b satisfying the equation $la+mb=x$. If we can have another piece of infor-

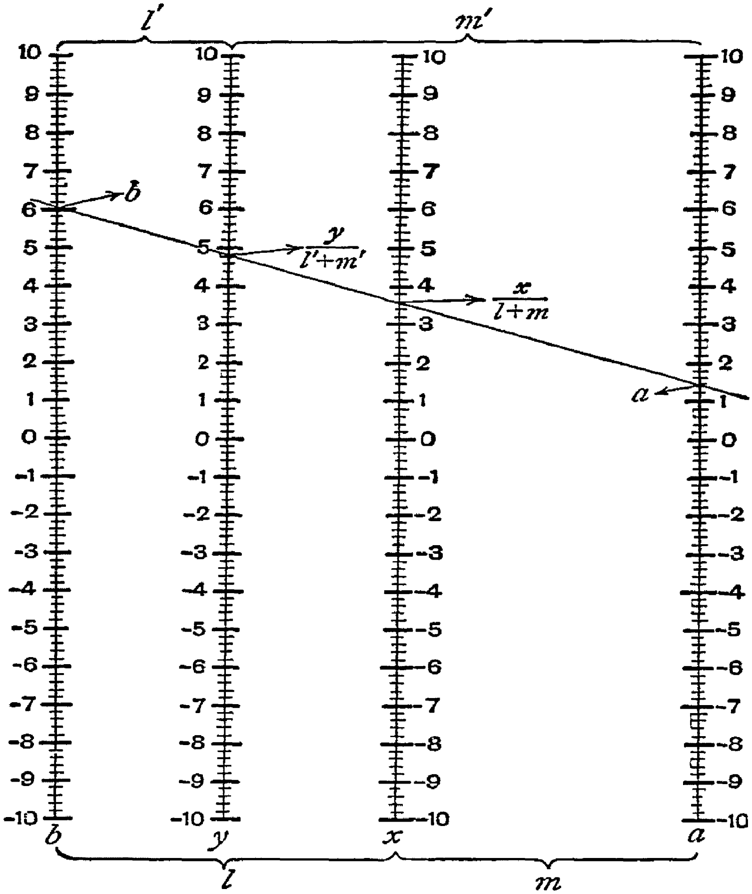


FIG. 9.

mation so as to decide upon the particular line through x to be used, we shall have unique values of a, b. This information can be supplied by another equation, $l'a+m'b=y$. Let us then introduce a fourth scale, y (Fig. 9), equal in all respects to the a, b, x scales, but dividing the distance between a, b in the ratio $m' : l'$, and let the y graduations be $l'+m'$ times the y distances. Then a line passing through a point y $\left(i.e.\text{ graduation } \frac{y}{l'+m'}\right)$ cuts the scales a, b at graduations satisfying the equation $l'a+m'b=y$.

If, therefore, we take the line through the point x $\left(\text{graduation } \frac{x}{l+m}\right)$ on scale x, and the point y $\left(\text{graduation } \frac{y}{l'+m'}\right)$ on scale y, it cuts the scales a, b at points whose graduations satisfy the simultaneous equations $la+mb=x,\quad l'a+m'b=y.$

In other words, we have a graphical solution of these simultaneous equations, in which x, y are given numbers and a, b are the unknowns.

Thus, in Fig. 7 we have $m : l=1$, $m' : l'=2$, so that Fig. 7 is a nomogram for the equations

$$a+b=x, \quad a+2b=y,$$

in which we use the graduations $x/2$ and $y/3$ on the x, y scales respectively. Also, in Fig. 8, $m : l=2 : 3$, $m' : l'=6 : 1$, so that we have a nomogram for

$$3a+2b=x, \quad a+6b=y,$$

in which we use graduations $x/5$ and $y/7$.

9. Nomogram for all Simultaneous Equations with Positive Coefficients

If we wish to construct a nomogram to be used for all positive values of l, m, l', m', we have to construct x, y scales for all ratios $m : l$, $m' : l'$. We have then the following method.

10. Segmentary Scale.

Let AB be a given line (Fig. 10), and let P be a point on this line between A and B. The position of P is given by the ratio $AP : PB$.

Let the value of $AP : PB$ be put at the side of the point P. If this is done for a (convenient) number of values of the ratio $AP : PB$, we have a segmentary scale, *i.e.* a scale in which the graduation at any

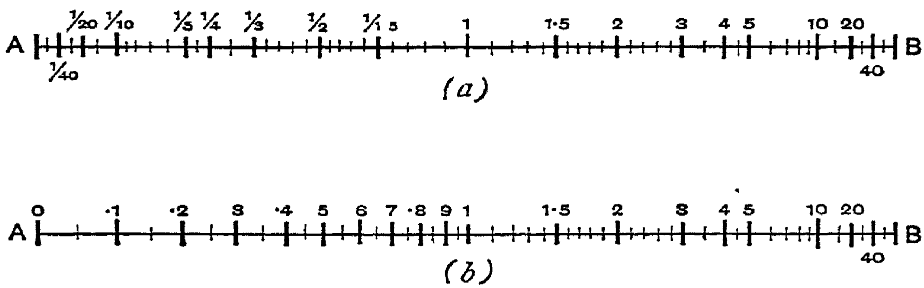


FIG. 10.

point gives the ratio of the segments. The mid-point is marked 1, on the right we have graduations greater than 1, on the left graduations less than 1. The segmentary scale can be graduated in one of two ways. Either the graduations less than 1 are given in the form of reciprocals, as in Fig 10 (a), or they are given in the form of decimals, as in Fig 10 (b). We shall find both forms useful. For simultaneous equations we use the form 10 (a).

11. Nomogram for Simultaneous Equations.

Take two equal parallel scales a, b; graduate the distance between them as a segmentary scale, 10 (a), and draw lines through the graduations parallel to a, b. Through the graduations of a, b draw lines parallel to the segmentary scale. We thus get Fig. 11, in which we have for any given segmentary ratio a scale graduated like a, b.

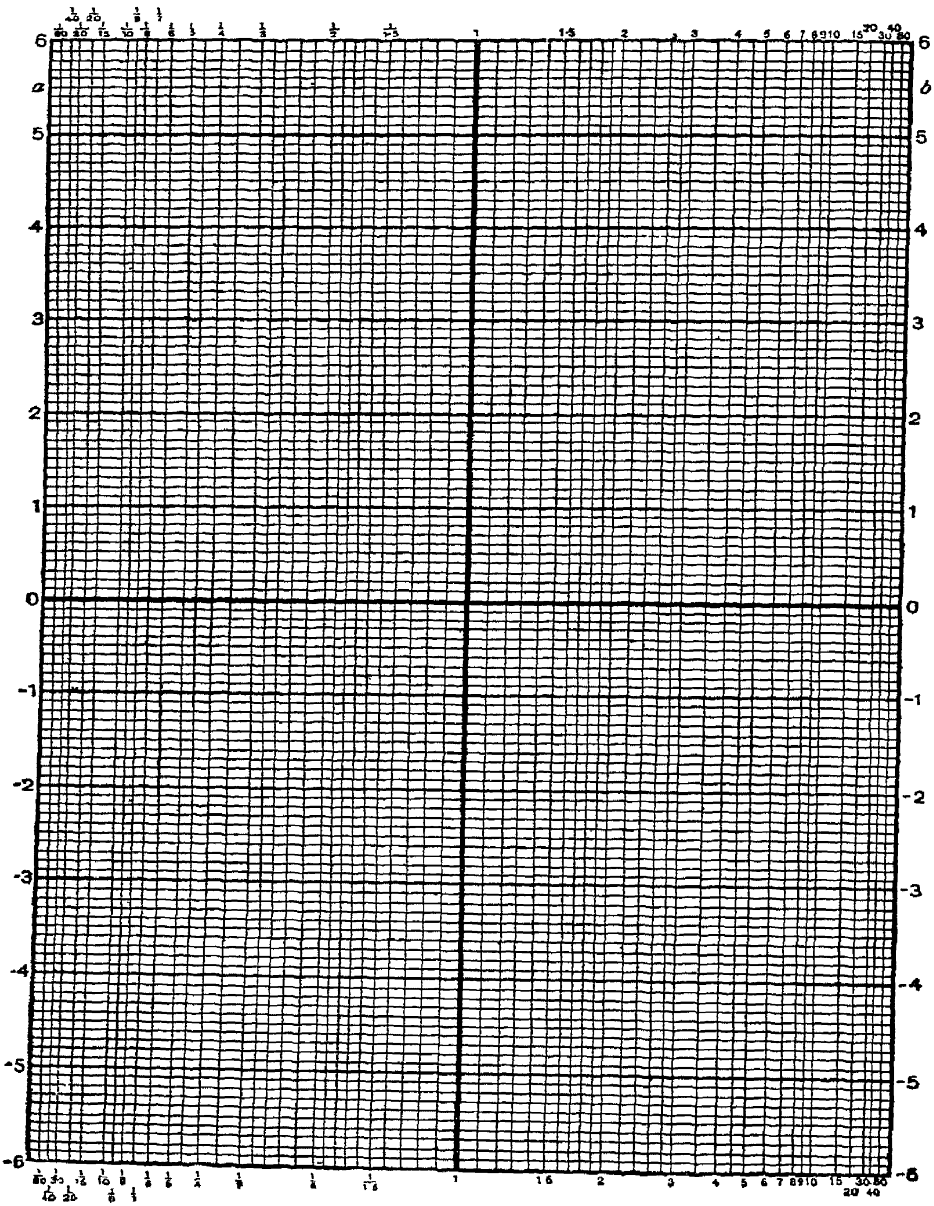


Fig 11.

To solve the equations $la+mb=x$, $l'a+m'b=y$, take the point $\frac{x}{l+m}$ on the scale $m:l$ (by interpolation at sight if necessary) and the point $\frac{y}{l'+m'}$ on the scale $m':l'$; the join cuts a, b in points giving the solutions required.

Thus, to solve the equations

$$3a+8b=23, \quad 4a+5b=18,$$

we take the point $\frac{23}{11}$ on the scale $8:3$, and the point $\frac{18}{9}$ on the scale $5:4$. The line joining the points cuts the a, b scales at $a=1{\cdot}71$, $b=2{\cdot}23$, which are the solutions of the given equations.

12. Simultaneous Equations with Negative Coefficients.

Fig. 11 can be used for all values of x, y, positive or negative (see § 5), and the solutions a, b will come out with their proper signs. We have, however, to solve equations in which the coefficients l, m, l', m' may be negative, whereas the nomogram in Fig. 11 is constructed for positive values of the ratios $m:l$, $m':l'$. But this nomogram can really be used always. We can assume l, l' both positive, since, if either is negative, we can introduce a negative sign all through the corresponding equation: *e.g.* $-a+3b=5$ can be written $a-3b=-5$. If now m, m' are *both* negative, we can put $b'=-b$, and we get two equations with positive coefficients. Thus, equations

$$a-3b=-5, \quad 2a-b=7$$

can be written

$$a+3b'=-5, \quad 2a+b'=7,$$

where $b' = -b$, and then we can use the nomogram of Fig. 11, calling b, b'. For the equations just given we take the point $-\frac{5}{4}$ on the scale 3 : 1, and the point $\frac{7}{3}$ on the scale 1 : 2, and we get $a = 5{\cdot}2$, $b' = -3{\cdot}4$, so that the solutions required are $a = 5{\cdot}2$, $b = 3{\cdot}4$.

If m, m' are not both negative, but one positive and the other negative, say m positive and m' negative, it is an easy matter to add a multiple of the first equation to the second so as to make the coefficients all positive. *E.g.*, to solve the equations

$$3a + 8b = 23, \quad a - 3b = -5,$$

we add once the first equation to the second, so that we get

$$3a + 8b = 23, \quad 4a + 5b = 18,$$

and we solve as already suggested (§ 11).

13. Extended Use of the Nomogram for Simultaneous Equations.

The nomogram of Fig. 11 will do more than merely solve for a and b. It will give us directly, without finding a and b, the value of $l''a + m''b$, it being given that $la + mb = x$, $l'a + m'b = y$. Find the point $\dfrac{x}{l+m}$ on the $m : l$ scale, and the point $\dfrac{y}{l'+m'}$ on the $m' : l'$ scale; let the join cut the scale $m'' : l''$ (assumed positive) and multiply the graduation obtained by $l'' + m''$: the result is the required value of $l''a + m''b$. Thus, to find the value of $2a + 6b$, where $3a + 8b = 23$, $a - 3b = -5$, we use the equations $3a + 8b = 23$, $4a + 5b = 18$; we take $\frac{23}{11}$ on the scale 8 : 3 and $\frac{18}{9}$

on the scale 5 : 4 ; the join cuts the 6 : 2, *i.e.* 3 : 1, scale at 2·1, giving, on multiplication by 8,

$$2a+6b=16{\cdot}8.$$

With negative values of l'', m'' we can proceed as in § 12.

Thus, $\quad -2a-6b=-(2a+6b)=-(16{\cdot}8)=-16{\cdot}8.$

To find the value of $2a-6b$, we use the fact that $3a+8b=23$, and put

$$(2a-6b)+(3a+8b)=5a+2b,$$

so that $\quad 2a-6b=5a+2b-23.$

We find $5a+2b$, and we get $2a-6b=-9{\cdot}96.$

EXAMPLES I.

Note.—Nomograms should be constructed on sheets of stout paper or cardboard of the usual quarto shape and size. The figure in each case should be as "square" as possible, *i.e.* the whole should be roughly contained within a square. At first, for practice, squared paper may be used. When some experience in the construction of diagrams has been obtained, it is better to use plain white sheets. In graduating the scales it is useful to adopt the following rules :

(i) No two subdivisions should be less than 1 mm. apart ;

(ii) Useful subdivisions are halves, fifths, tenths, twentieths, fiftieths, etc. ;

(iii) Where changes in the manner of subdivision are introduced, they should be at well-marked positions, as *e.g.* at the ends of whole numbers.

It will be a profitable exercise to construct the segmentary scales 10 (*a*), (*b*) with units 15 cm , 20 cm , 30 cm. respectively, putting in all the useful subdivisions in each case Fig. 10 (*a*), (*b*) is drawn for an 8 5 cm unit, and should be studied in the light of the rules just given.

1. Construct nomograms for

(i) $2a - b$; (ii) $a - 2b$; (iii) $1{\cdot}7a + 2{\cdot}3b$;

(iv) $a - \frac{1}{2}b$; (v) $-\frac{2}{3}a + \frac{1}{2}b$; (vi) $3a + 3b$.

2. Solve by means of a nomogram the equations

$$\tfrac{3}{2}x - 2y = 5; \quad x - 3y = -7.$$

Deduce the values of $7x + y$; $2x - 5y$; $-x + 2\frac{1}{2}y$.

3. Construct a segmentary scale with 10 cm. unit, and extend it on both sides so as to include negative graduations from 0 to $-0{\cdot}5$ and from -2 to ∞ respectively. Insert all the useful subdivisions.

4. Construct a nomogram for $la + mb = nc$, it being given that $a + b + c = 100$, to be used for all positive values of l, m, n.

CHAPTER II

GENERALISED NOMOGRAMS FOR ADDITION AND SUBTRACTION

WE have seen how the simple idea with which we started off in Chapter I, § 3, can be applied to the construction of a nomogram for comparatively complicated calculations. Before proceeding to the application of the idea to successive additions and subtractions we shall consider one or two modifications in the method already described.

14. Nomograms with Equidistant Scales.

Compare Figs. 4 and 5. It has already been pointed out that in the latter we have two changes—the scale x is displaced and differently graduated. For the purpose of simultaneous equations it has been found useful to keep the x graduations invariable for all values of l, m in $la+mb=x$; in fact the x graduations are made the same as the a, b graduations (§ 7) For purposes other than the solution of simultaneous equations it is an advantage to arrange that the x scale shall have a stationary position, since the finding of the position of the x scale involves finding the segmentary division $m:l$, and this introduces some error. We shall now see that we can arrange to have the x scale not only fixed in position, but also graduated in a fixed manner. The method is shown in Fig. 12.

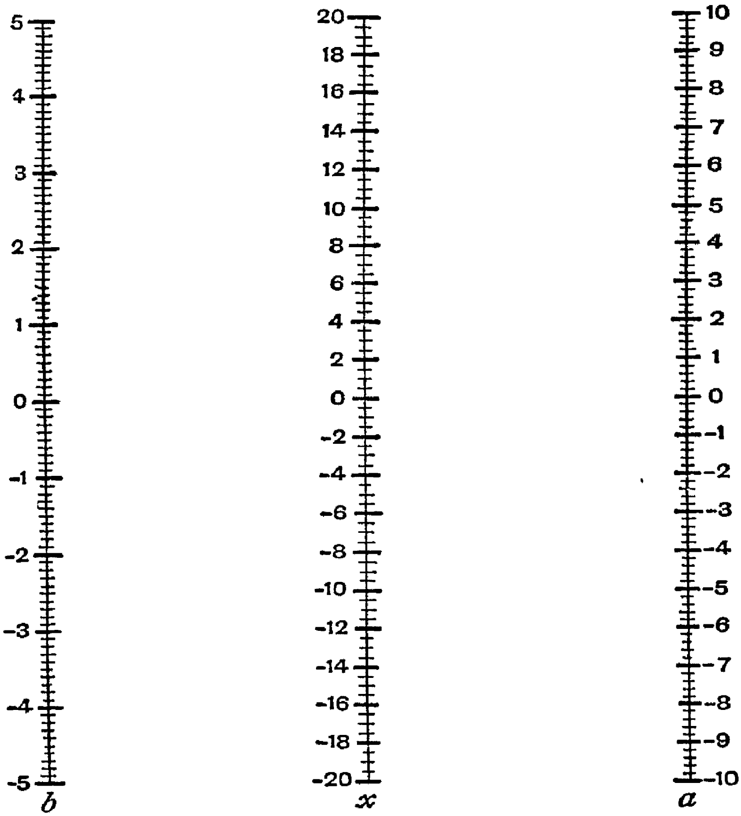


FIG. 12.

Let the scales a, x be as in Fig. 4, but graduate the b scale with a unit double that of a, so that we have three parallel scales a, x, b; x midway between a and b, the unit of a being twice that of x, and the unit of b being four times that of x. Now, if a straight line cuts the scales at a, x, b, we have

twice distance $x =$ *distance* $a +$ *distance* b.

It follows that

graduation x = *graduation* a + *twice graduation* b.

In algebraic symbols we have

$$x = a + 2b.$$

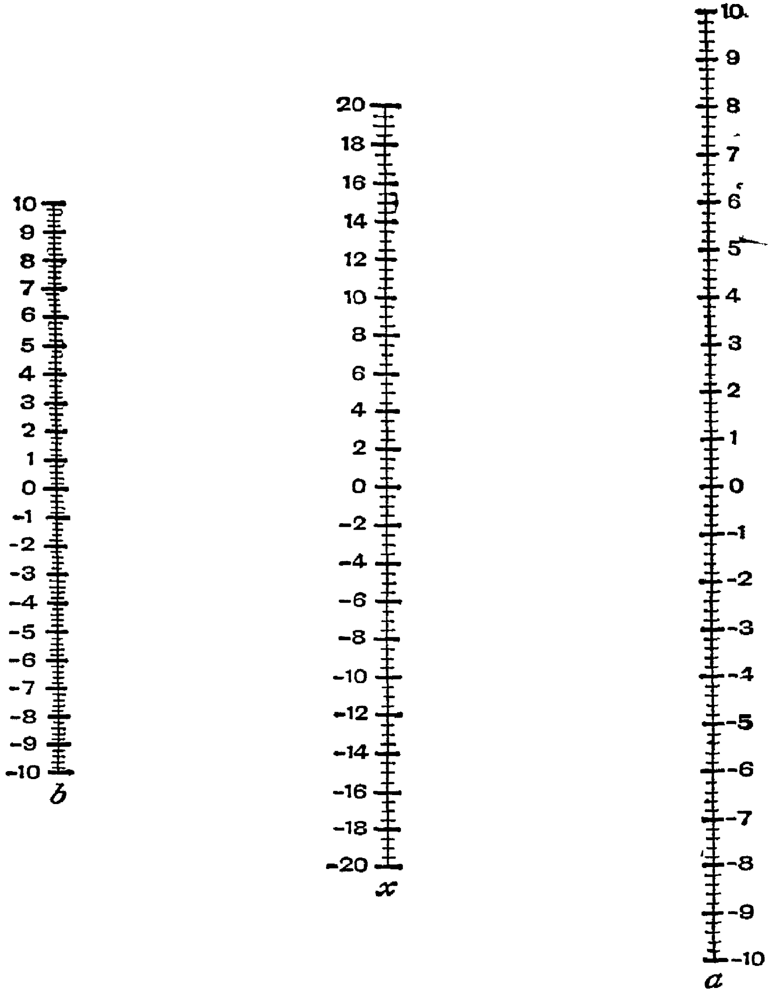


FIG 13.

We thus have another nomogram for $a+2b$, which is preferable for most purposes to that of Fig. 5.

The method can be extended to

$$x = la + mb$$

for all values of l, m. Take three parallel scales a, x, b; make the a unit $2l$ times the x unit, and the b unit $2m$ times the x unit.

The geometrical fact that three collinear points a, x, b give

twice distance x = *distance* a + *distance* b,

now becomes

graduation $x = l \times$ *graduation* $a + m \times$ *graduation* b;

algebraically, $$x = la + mb.$$

Thus, when $x = \frac{5}{4}a + \frac{3}{4}b$, we have $l = 1\frac{1}{4}$, $m = \frac{3}{4}$, so that the a scale is made with a unit $2\frac{1}{2}$ times the x unit and the b scale with a unit $1\frac{1}{2}$ times the x unit (see Fig. 13).

15. Subtraction, Simple and Extended.

We can now introduce negative coefficients. Suppose we require a nomogram for $x = a - b$. Looking upon this as a case of $x = la + mb$, in which $l = 1$, $m = -1$, the way to make such a nomogram is at once suggested. Take three equidistant scales a, x, b, the x scale being midway, and graduate the a scale with twice the x unit, and the b scale with *minus twice* the x unit, *i.e.* with twice the x unit but with the *signs of the graduations reversed*. For collinear points a, x, b we have (Fig. 14)

twice distance x = *distance* a + *distance* b, so that

$$\textit{graduation } x = \textit{graduation } a - \textit{graduation } b,$$

i.e. algebraically $x = a - b.$

The reader is advised to note this special case with particular attention, as it will play an important part in what follows.

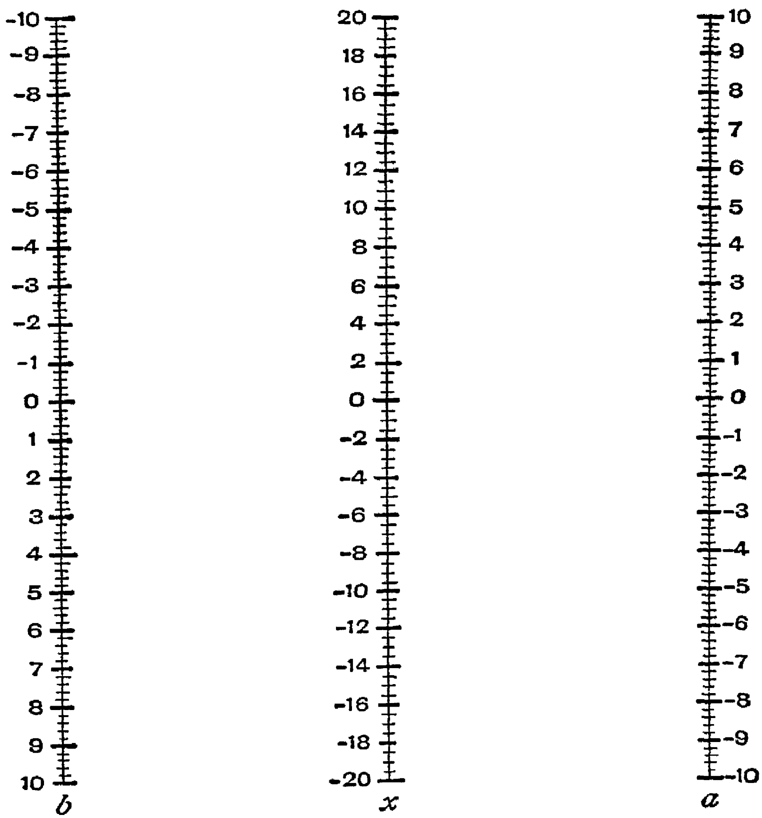


FIG 14

More generally, if in $x = la + mb$, one or both of the coefficients l, m are negative, we reverse the signs in the graduations of the corresponding scale. Thus, in Fig. 15 we have the nomogram for

$$x = a - 2b,$$

being in fact the same as that given in Fig. 12 for

$$x = a + 2b,$$

but with the signs of the b graduations reversed.

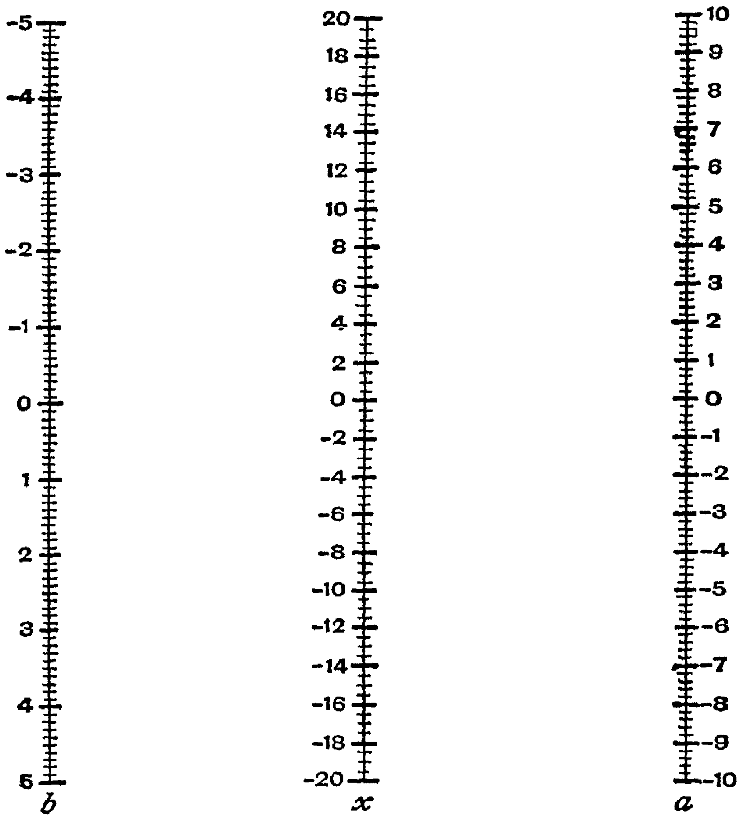


FIG. 15.

16. Generalised Addition with an Added Constant.

We have to note one more respect in which th nomogram for the addition of two quantities need to be extended. This arises in the case when *constant* quantity has to be added on to the result obtained by adding two numbers which can assume any values. Let us consider

$$x = a + b + 1.$$

This can be written

$$x = (a + 1) + b,$$

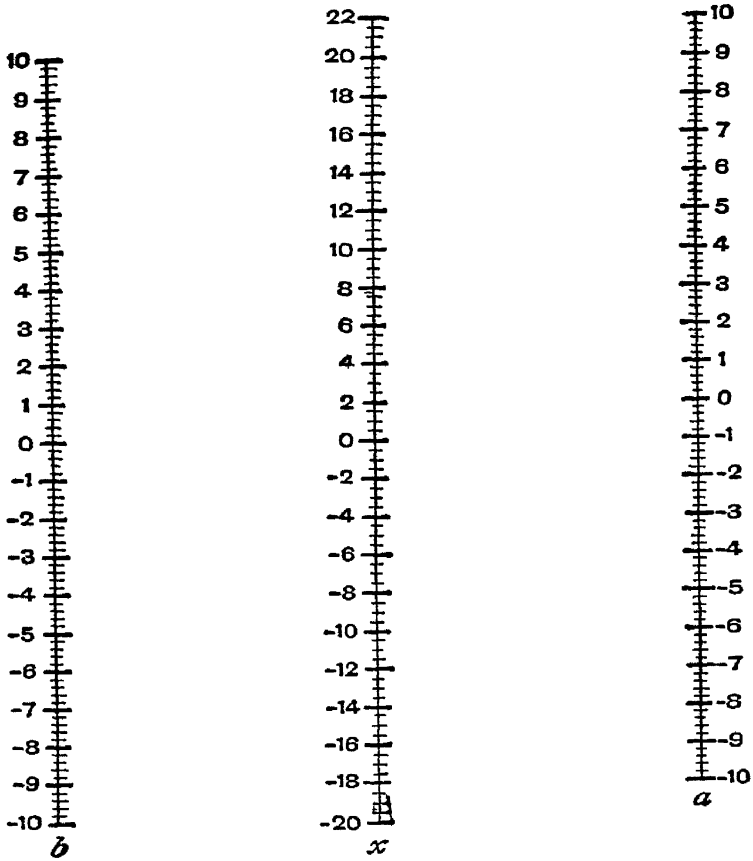


FIG 16

which means that each point on the *a* scale of the nomogram for $x = a + b$ should be graduated *one a unit less*, as in Fig. 16. (It need hardly be pointed out that the same result is obtained by modifying the b scale instead of the a scale, or by modifying both in such a way that the total effect is to add on 1 to the value of $a+b$ as given by the graduations on collinear points. In any particular case circumstances and discretion will decide the best course to adopt. The same applies to other such examples in the sequel.)

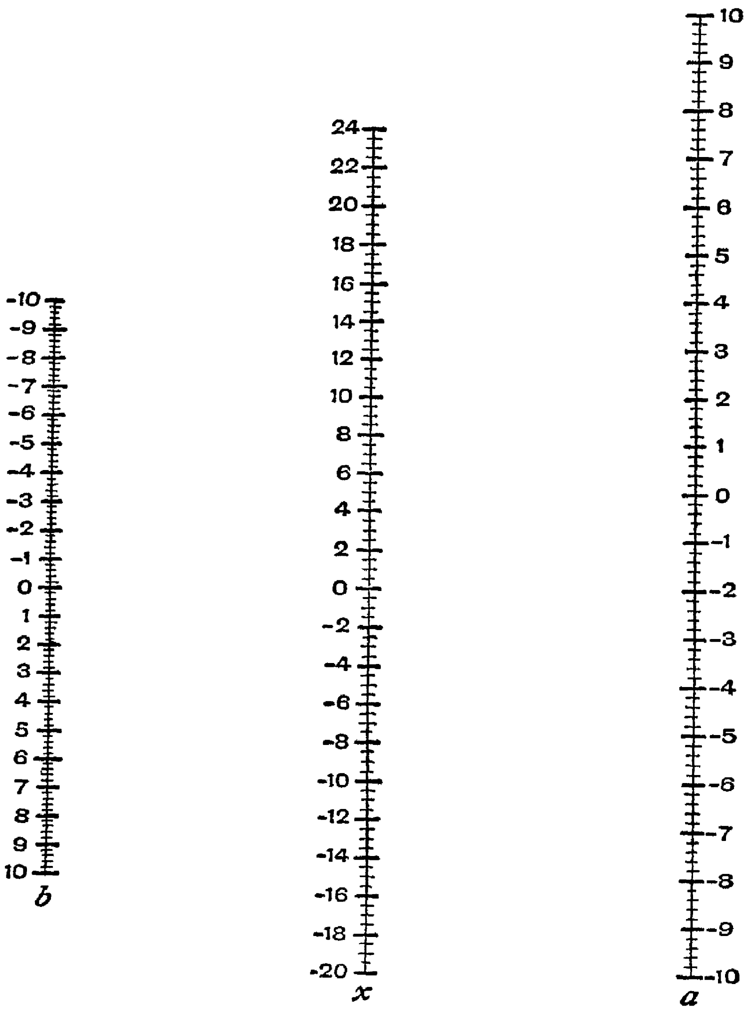


FIG. 17.

17. Generalising at once we see that to obtain a nomogram for $x = la + mb + n,$

take three parallel scales a, x, b; x being midway between a and b; take the zero of x, the graduation $-n/l$ of a, and the zero of b collinear; graduate a with unit $2l$ times that of x (signs reversed if l is negative), and graduate b with unit $2m$ times that of x (signs reversed if m is negative). Then three collinear points a, x, b give

twice distance x
$\quad = $ *distance a + distance b +* $2n$ *units of x,*

i.e. *twice no. of x units*
$\quad = 2l($*no. of a units*$) + 2m($*no. of b units*$)$
$\quad + 2n($*no. of x units*$),$

i.e. *graduation x*
$\quad = l \times$ *graduation a* $+ m \times$ *graduation b* $+ n,$

or algebraically $x = la + mb + n.$

In Fig. 17, we have taken $l = 1\frac{1}{4}$, $m = -\frac{3}{4}$, $n = 2\frac{1}{2}$.

18. Successive Addition

We have so far considered only the case of two added quantities la, mb, with a possible added constant. It is now required to extend the method to three and more added quantities. Let us begin, as before, with a special case. Take

$$x = a + b + c.$$

The method that suggests itself is to add a and b by the nomogram of Fig. 4, and then to add $(a + b)$ and c by another such nomogram. Let us put

$$y = a + b, \quad x = y + c.$$

If we use the nomogram of Fig. 4, we must have the y scale half way between a and b, and the x scale half way between y and c. Also the three scales a, b, c must be quite distinct. The most compact arrange-

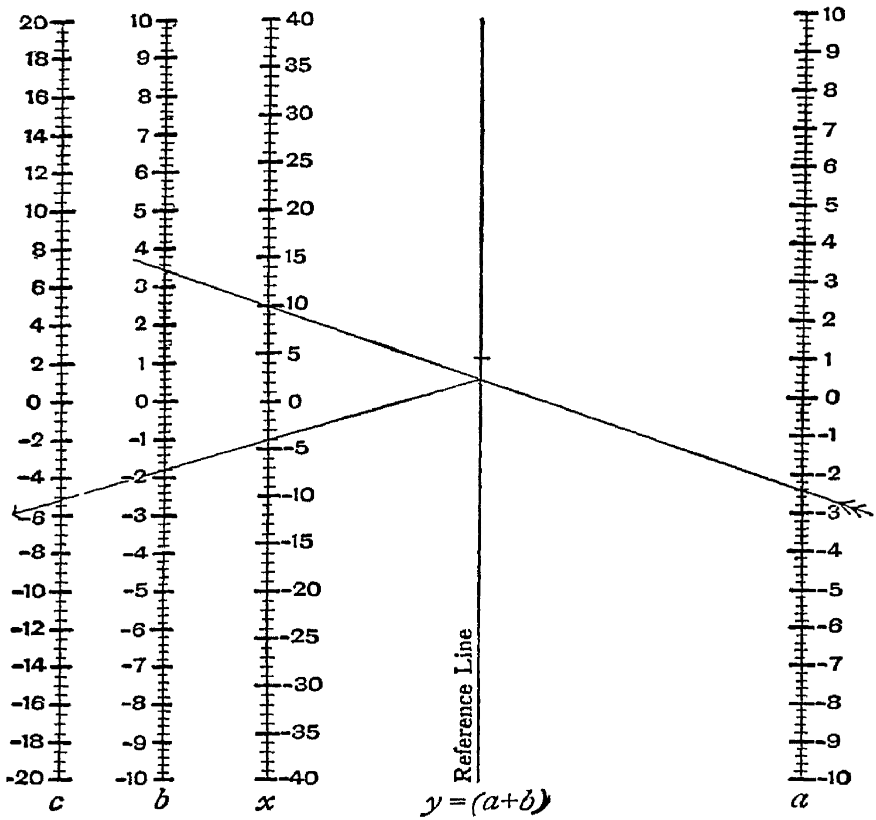


FIG. 18.

ment is shown in Fig. 18. Further, the y unit is half of the a or b unit, and the c unit must therefore also be half, and the x unit a quarter of the a, b unit. Fig. 18 is therefore obtained as follows. Take three parallel lines a, y, b at convenient equal intervals, and having graduated a, b equally, graduate y with half the a, b unit. Now take parallel lines c, x, so that x is midway between y and c, and the distance of x from y is, say, two thirds that of b from y. Graduate c in the same way as y, and x with half the y unit, *i.e.* with a quarter of the a, b unit. Take the point a on the a scale and the point b on the b scale: the join cuts the y scale at the graduation $a+b$; join this point to the point c on the c scale; the line thus obtained cuts the x scale at the graduation $a+b+c$.

One fact is immediately made evident. *It is not necessary to graduate* y *at all*, since the actual value of $a+b$ is not really required—only the *position* of the $a+b$ graduation on the y line is wanted. We therefore leave the y line—called a Pivotal or **Reference Line**—undivided. Further, we must indicate where the operation begins. We therefore draw cross lines as shown in Fig. 18, the arrow-feathers showing where to begin, and the bending of the cross lines where the reference line is crossed.

But it is further clear that the successive application of the addition nomogram must soon lead to impracticable diagrams. The more obvious difficulty is the successive diminution in the units, which, if allowed to continue, must result in inconveniently fine graduations or uselessly gross and inaccurate readings. The other difficulty is that in order to keep the various scales distinct, the width of the nomogram will, after a few additions, become unmanageable.

To overcome the first difficulty, namely the diminution in the units, we must evolve a *nomogram for* *addition in which the result of the addition is on a scale with undiminished unit.*

This is at once obtained by means of the nomogram for simple subtraction (Fig. 14). Referring back to Fig. 14, we see that if the a, b graduations are equal but in opposite directions, and the x graduations are with half the a unit in the same direction as the a graduations, then three collinear graduations a, x, b give

$$x=a-b, \quad i.e. \quad a=b+x.$$

Hence the sum of two quantities b, x is given in the same unit as that used for one of these quantities, namely b. Hence we get a new and

19. Alternative Nomogram for Addition.

Rule I To find $\mathbf{x=a+b}$

take three parallel lines a, b, x; b being midway between a and x. Graduate a with a convenient unit, then x with the same unit in the opposite direction; finally graduate b with half the a unit, and also in the opposite direction, the three zeros being collinear Three collinear graduations will then satisfy the given equation (Fig. 19).

We shall use this alternative nomogram for addition in all cases where the process of addition has to be carried out

Incidentally it will be seen that since the result of the addition is on one of the exterior lines, the successive application of this nomogram will lead to more compact diagrams.

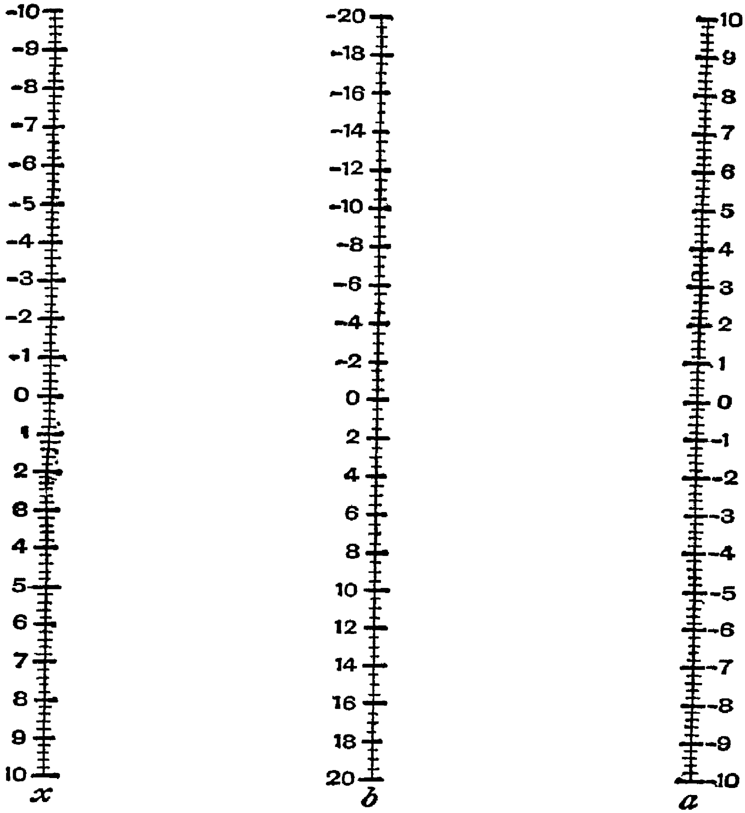


FIG 19.

20. Nomogram for a + b + c.

We proceed to illustrate the use of the alternative method of the last article by application to the case

$$x = a + b + c.$$

In Fig. 20 take a scale *up* and scale b at *half unit down*: we get $a+b$—reference line—at *full unit down, but not graduated.* Now take scale c at *half unit up*, in some convenient position between $a+b$ and b: we get $a+b+c$ at *full unit up.* The cross lines show the order of the process and the starting point is indicated by the arrow-feathers.

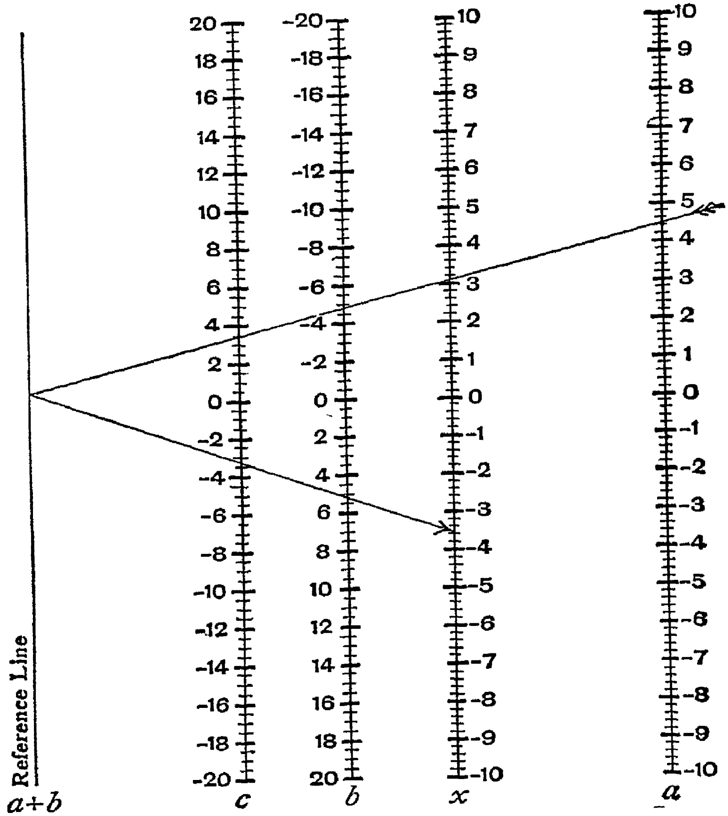


FIG 20.

21. Nomogram for x = a + b + c + d.

We are now in a position to construct a more extended nomogram, as *e.g.* for

$$x = a + b + c + d.$$

In Fig. 21, take *a* scale *up*, scale *b* at *half unit down*: we get reference line (i), which is really $a+b$ at *full unit down*, but not graduated. Now take scale *c* at *half unit up*, midway between *a*, *b*: we get reference line (ii), which is $a+b+c$ at *full unit up.* Take scale d at *half unit down, midway* between a, c: we get $x=a+b+c+d$ at *full unit down.* A preliminary sketch will at once indicate at what distance apart

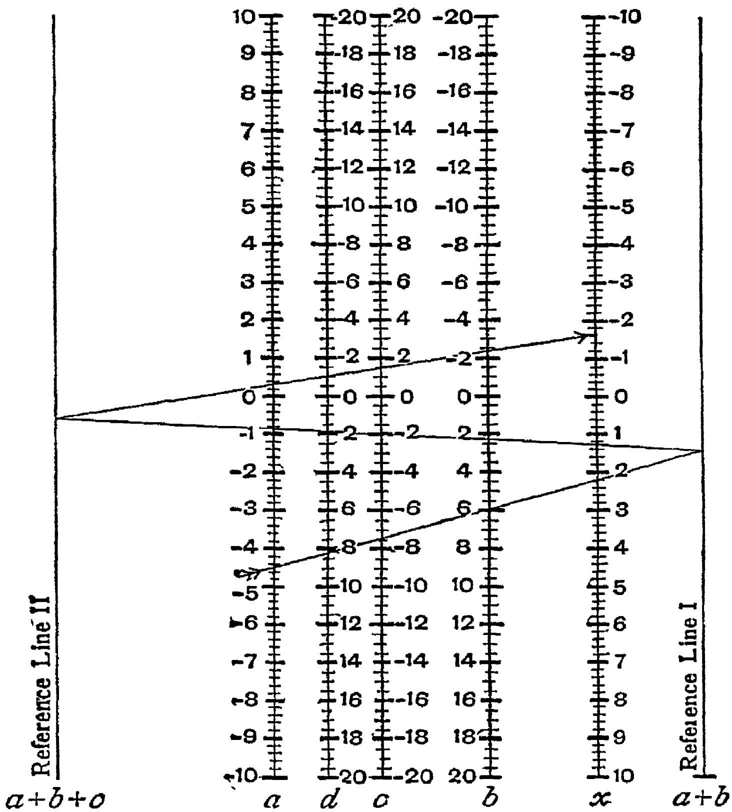


FIG. 21.

to take a, b initially in order that we may get nicely spaced scales. The cross lines and the arrow-feathers indicate the order of the process and the reference lines.

22. Subtraction.

The question now arises: What process shall we adopt when we have both addition and subtraction in one nomogram? There are several processes for subtraction: thus Fig. 4 gives $b=x-a$, and Fig. 14 gives $x=a-b$. If we examine these figures we shall see that, in subtracting, the final unit is bound to be different from one or other of the quantities whose difference is being found. It follows that in doing additions and subtractions care must be taken to avoid the successive growth or the successive diminution of the final unit. In order to render it unnecessary to waste time in the taking of this precaution in any actual case, it is therefore advisable to adopt a definite rule, which can be followed without regard to the special necessities of the case. The method to be adopted will perhaps be most easily grasped by considering a simple case.

23. Nomogram for $a+b-c$.

Instead of leaving the subtraction to the end, we write the operations in the form

$$x=(a-c)+b,$$

so that we shall subtract at once. In Fig. 22 we have a with *full unit up*, c with *full unit down*, so that the reference line $(a-c)$ is obtained midway and with *half unit up*, but ungraduated. Now take b with *full unit down* on one side of the reference line: we get $x=a-c+b$ with *full unit up* at an equal distance on the other side of the reference line.

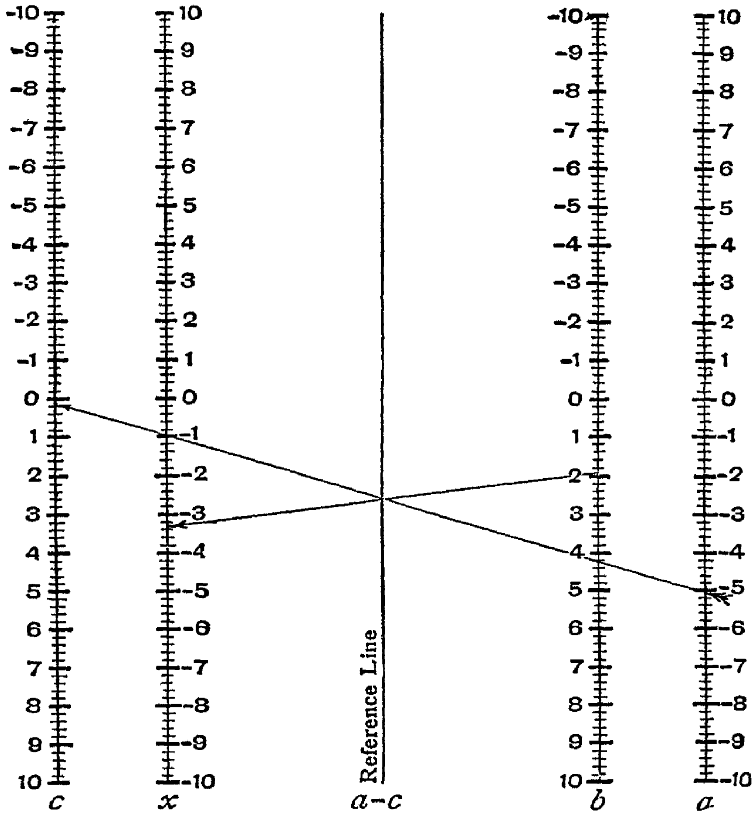


FIG. 22.

24. Successive Subtraction.

A little thought will convince one that for any type of operation involving additions and subtractions, there are a number of different nomograms possible Without claiming for the general rule to be given here the title of being necessarily the best under all conditions, its adoption is suggested as a means of introducing unity and compactness of treatment in this book The reader can introduce variations at his discretion

Rule II When an operation involves a number of additions and subtractions begin by adding up nomographically all the negative terms, subtract nomographically the sum from the first positive term, and then add nomographically the remaining positive terms: each addition is to be performed by means of the alternative method (§ 19).

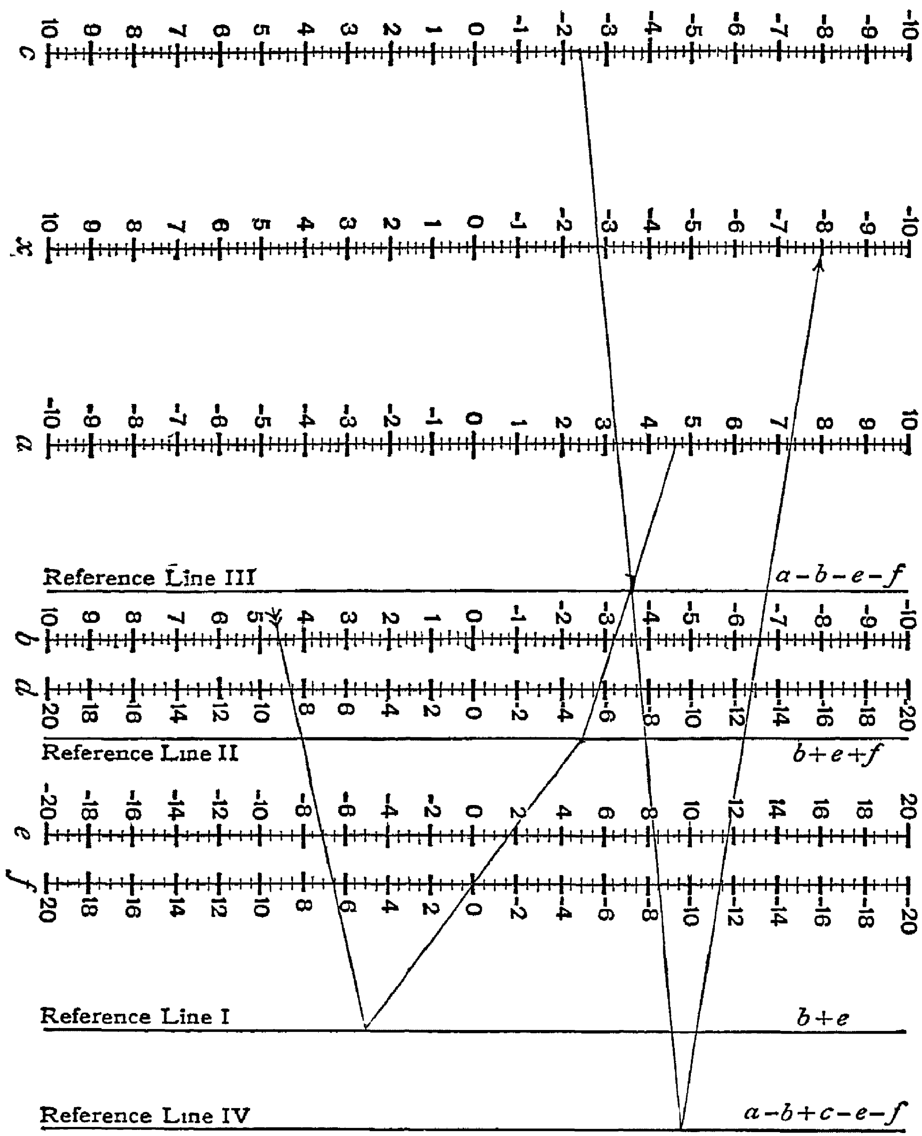


FIG. 23.

25. Nomogram for a − b + c + d − e − f.

We write this in the form

$$x=a-(b+e+f)+c+d,$$

and first find a nomogram for $b+e+f$. In Fig. 23 we have b with *full unit down*, and e with *half unit up*, giving us the reference line (i), which is $b+e$ at *full unit up*, but ungraduated. We take f with *half unit down*, and we get reference line (ii) (ungraduated), which is $b+e+f$ with *full unit down*, the position of f being so chosen that this reference line comes into a convenient position, say midway between b and e. Now take a with *full unit up*, giving reference line (iii) (ungraduated), which is $a-(b+e+f)$ with *half unit up* (in the figure a is placed so that this reference line falls at a convenient distance to the left of b). If we take c with *full unit down*, we get reference line (iv) for $a-b+c-e-f$ with *full unit up*, and finally, taking d with *half unit down*, we get the final scale for

$$x=a-b+c+d-e-f$$

with *full unit down*.

The order of the processes is sufficiently indicated by means of the cross lines, the arrow-feathers showing the starting off line.

26. Generalised Nomogram for Addition and Subtraction.

The most general form that a series of additions and subtractions can assume is

$$x=la+mb+nc+\ldots-pf-qg\ldots+s \text{ (or } -s),$$

where l, m, n, p, . . . , s are *positive constants*. At the stage we have now reached we can construct a nomogram for any such expression. The method will be illustrated sufficiently by means of a special case. Let us then take

$$x = 15a + b - 7{\cdot}5c - \tfrac{2}{3}d + 10e + 4.$$

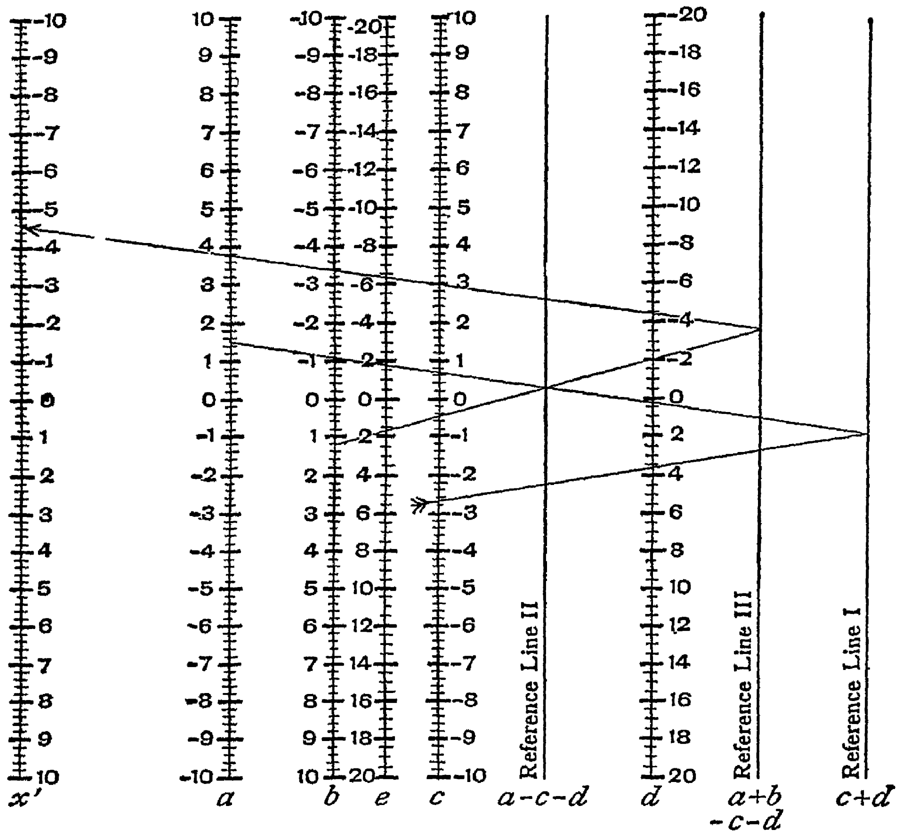


FIG. 24.

We first work out a nomogram for

$$x' = a + b - c - d + e,$$

since we know that the coefficients and the added constant can be taken into account in the graduation of the scales. Fig. 24 gives such a nomogram, the

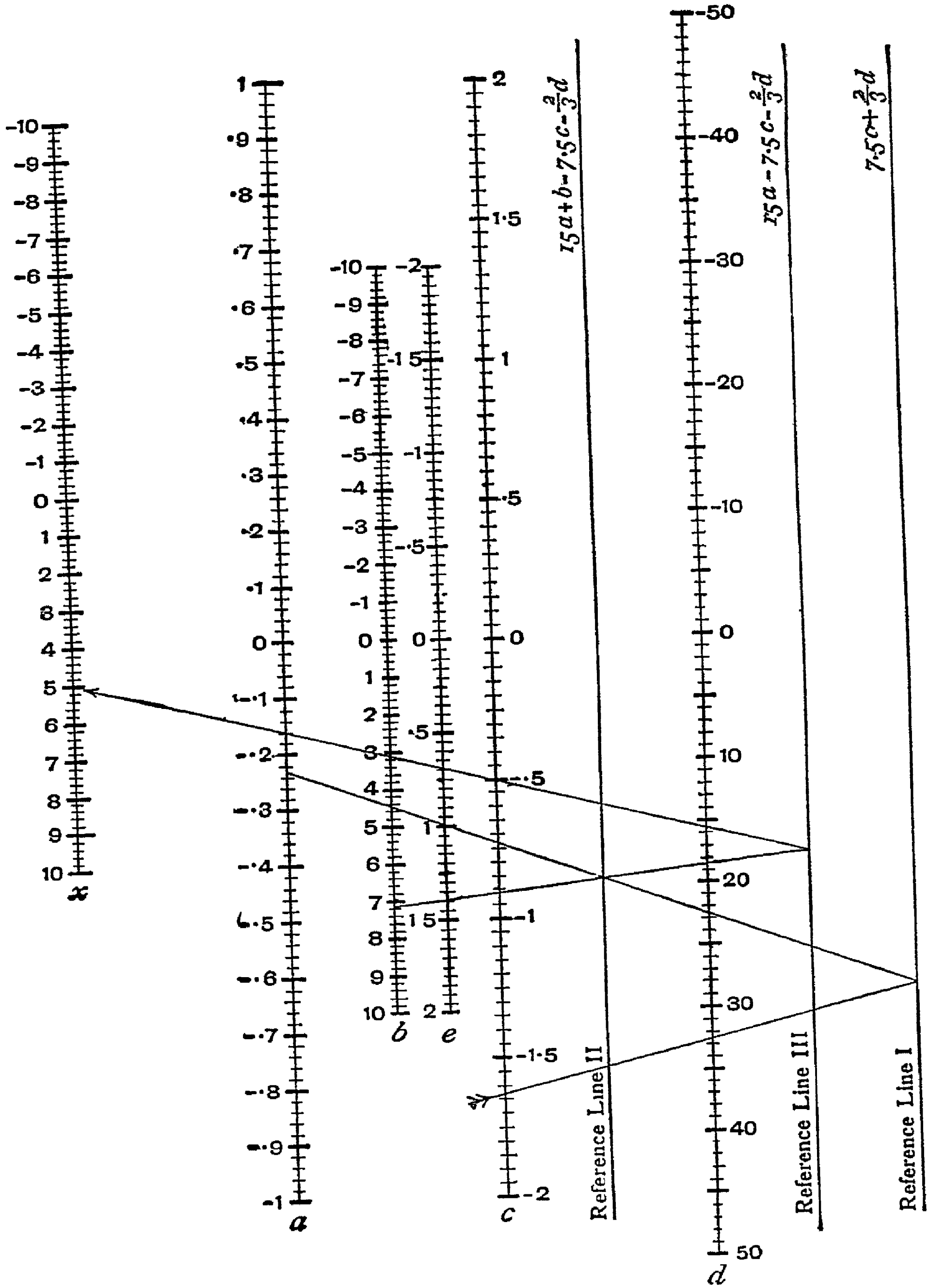

x
a
b
e
c
d
$1{\cdot}5a+b-7{\cdot}5c-\frac{2}{3}d$
$1{\cdot}5a-7{\cdot}5c-\frac{2}{3}d$
$7{\cdot}5c+\frac{2}{3}d$
Reference Line I
Reference Line II
Reference Line III

FIG. 25.

analysis of which is left as an exercise to the reader. We now proceed to construct the graduations. We take (Fig. 25) the final scale to be with *full unit.* Remembering the rule of § 17, we therefore graduate a with unit fifteen times that in the preliminary nomogram already obtained, *i.e.* with unit 15 *times* the *full unit*; b with the *full unit,* since it must be the same as in the preliminary nomogram; c with 7·5 *times full unit*; d with $\frac{2}{3}$ of the unit in the preliminary nomogram, *i.e.* $\frac{1}{3}$ of the *full unit*; and e with 10 times the unit in the preliminary nomogram, *i.e.* with 5 times the *full unit.* The added constant can now be put in at some convenient stage, say in x itself, which is therefore made to have the graduation 4 where it would have had the graduation 0.

Rule III. To construct a nomogram for

$$x = la + mb + nc + \ldots - pf - qg \ldots + s \text{ (or } -s)$$

$l, m, \ldots, s$ being positive, first construct a preliminary nomogram for

$$x' = a + b + c + \ldots - f - g \ldots$$

in which each term has the same sign as the corresponding term in x. Then multiply the a unit of the preliminary nomogram by l, the b unit by m, etc. Finally put in the added or subtracted constant $\pm s$ at some convenient stage.

Examples II.

1. Construct a nomogram for $2a + 3b$ using the alternative method, § 19.

2. Construct a nomogram for $2a - 3b + 4$.

3. Construct nomograms for

(i) $a+2b+3c$; (ii) $a-2b+3c$;
(iii) $a+2b-3c$; (iv) $a+2b+3c-4$;
(v) $a-2b+3c+5$; (vi) $a+2b-3c-2\frac{1}{2}$.

4. Construct a nomogram for $-a+b+c$.
Convert it into one for $a-b-c$, and then into one for

$$2a-\tfrac{1}{2}b-4c+3.$$

5. Construct a nomogram for $\dfrac{1}{a}+\dfrac{1}{b}=x$, in which a and b can have values lying between 5 and 15.

Note.—Let $a'=1/a$, $b'=1/b$. Then $x=a'+b'$. Construct the nomogram for this and then convert the a', b' graduations into a, b graduations.

6. Construct a nomogram for $\dfrac{3}{a}-\dfrac{2}{b}=\dfrac{1}{x}$.

7. Construct a nomogram for $\dfrac{1}{f}=0{\cdot}517\left(\dfrac{1}{r}-\dfrac{1}{s}\right)$.

8. Convert the nomogram for $a+b=c$ into one for $a^2+b^2=c^2$; this will give the diagonal of a right-angled triangle in terms of the sides.

9. Construct nomograms for

(i) $v=u+32t$; (ii) $v^2=u^2-64s$; (iii) $v^2=u^2+64s$.

10. Construct a nomogram for $k^2=\frac{1}{3}(a^2+b^2+c^2)$.

CHAPTER III

NOMOGRAMS FOR MULTIPLICATION AND DIVISION

If the reader has made himself familiar with the methods and results of Chapters I. and II., he will be able to proceed to the construction and use of nomograms for processes involving any number of multiplications and divisions by factors raised to any powers. It is necessary to begin with an account of the

27. Logarithmic Scales.

It is probably safe to assume that everybody who is interested in practical calculations is familiar with the scales *A*, *B*, *C*, *D* of the ordinary slide rule. Thus

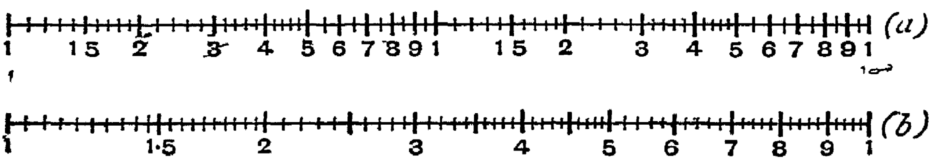


Fig. 26.

scales *C* and *D* are as shown in Fig. 26 (*b*), but on a larger scale. The *distance* of any graduation from the beginning of the scale is *proportional to the logarithm* of the number of the graduation. Hence the beginning of each scale is marked unity and the end should be marked ten. Since, however, powers of ten do not affect the mantissa, *i.e.* the decimal-fractional part of a logarithm, it is usual to mark the end of each scale unity—we can, in fact, consider the scale to refer to numbers 10^n to 10^{n+1}, where n is a positive or negative whole number.

To construct a logarithmic scale we take a uniformly graduated line, as *e.g.* on a sheet of squared paper, and mark off the logarithms as given in the Tables—this is sufficiently accurate for our purpose.

We first put in the numbers 2, 3, . . . , 9. Then we estimate how many subdivisions to include between 1 and 2, 2 and 3, and so on, remembering as mentioned in Chapter I., Note, that

(i) Graphically it is of little use to deal with excessively small intervals; in fact, the smallest subdivisions must not be much less than 1 mm., or $\frac{1}{25}$ inch.

(ii) The whole of any one interval 1 to 2, 2 to 3, . . . must be subdivided in the same way.

(iii) The useful subdivisions for facility in reading are halves, fifths, tenths, twentieths, etc.

(In addition to the exercise in Ch. I. Note, it would be a very instructive exercise for the reader to study the slide rule scales from this point of view: in the ordinary rule the *C*, *D* scales are each 25 centimetres long, whilst in the *A*, *B* scales a 1–10 interval is just half, *i.e.* 12·5 centimetres long. He will find that the smallest subdivisions are a little more than $\frac{1}{2}$ mm.—this is because the graduations are carried out with special care on material specially chosen for the purpose.)

Note.—For practical purposes it is useful to remember the following simple method for obtaining the main graduations in a logarithmic scale. Having chosen 10 convenient units—say inches, or centimetres, or halves—then the numbers 2, 4, 8 are given by 3, 6 and (slightly more than) 9 units from the beginning. The number 5 is given by 3 units back from the end ; 3 by very nearly $4\frac{3}{4}$ units from the beginning and 9 by a shade less than $\frac{1}{2}$ unit from the end. We thus have the numbers 1, 2, 3, 4, 5, 8, 9, 10. The number 6 is got by adding 3 units to the graduation 3, and 7 by simple interpolation. The subdivisions $1\frac{1}{2}$, $2\frac{1}{2}$, $3\frac{1}{2}$ and $4\frac{1}{2}$ are given by taking 3 units back from the graduations 3, 5, 7, 9 respectively ; the other half integers by interpolation. Other subdivisions can also be derived in this way. A little practice will enable one to remember this method and to produce expeditiously sufficiently accurate logarithmic scales for the purposes of nomography. A good check is obtained by the consideration that the intervals corresponding to equal numerical differences must continually diminish from the beginning to the end of the scale.

The fundamental principle in the use of a logarithmic scale is that if we add up the geometrical distances corresponding to two numbers we get the geometrical distance for the product. This is the basis of the slide rule process for multiplication and division by the use of the scales C and D.

But scales C and D are each only one interval, say 1 to 10, or 10 to 100, or in general 10^n to 10^{n+1}, where n is a positive or negative whole number. If wider ranges are required, we need to use, say, two such intervals, corresponding to 1 to 100, 10 to 1000, or generally 10^n to 10^{n+2}. This is provided by the A, B scales, which are given in Fig. 20 (a) to compare with

C, *D*. If still wider ranges are desired still more intervals may be provided. For ordinary purposes the *A*, *B* scales in the slide rule are found sufficient, since we treat each factor that we have to use as being between 1 and 10, and decide on the position of the decimal point in the final result by common sense or by means of a very rough check.

28. We now proceed to convert the addition and subtraction nomograms into multiplication and division nomograms, by means of converting the regular scales of Chapters I. and II. into corresponding logarithmic scales.

29. Nomogram for Simple Multiplication and Division.

In Fig. 4 let us convert a 0 to 1 interval on a regular scale into a 1 to 10 logarithmic scale, designated by a capital letter. We thus get Fig. 27, in which the *A*, *B* scales are exactly similar and range from, say, 1 to 10, and the *X* scale is midway, and graduated with *half unit*, so that it ranges from 1 to 100. If now we have three collinear graduations *A*, *X*, *B*, we have, if the 1 graduations are collinear,

$$\textit{twice distance } X = \textit{distance } A + \textit{distance } B,$$

so that, as *X* is graduated with *half unit* of *A*, *B*, we have

$$\textit{log graduation } X = \textit{log graduation } A + \textit{log graduation } B,$$

i.e. $X = AB, \quad \text{or} \quad B = X/A, \quad A = X/B;$

so that we have a nomogram for multiplication and division that can be used for values of *A*, *B* ranging from 1 to 10, and of *X* from 1 to 100.

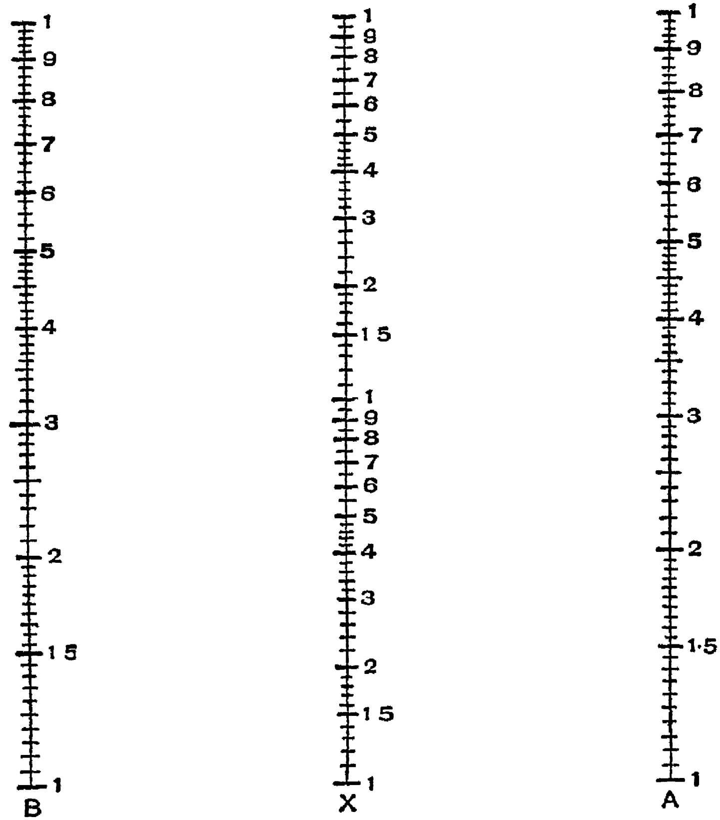


FIG. 27

30. Alternative Nomogram for Multiplication and Division.

As has been explained in the case of addition (Ch. II., § 18), it is not desirable to use the fundamental nomogram, especially when successive multiplications have to be carried out. We therefore introduce an alternative method, which is obtained from the alternative nomogram for addition, Fig. 19.

Rule IV. To make a nomogram for

$$X = AB,$$

we take A and X with the same unit, but with the graduations in X in the reversed order to that in A, and then put in

B midway with half unit, in the same order as X, i.e. also in the reversed order to that in A. We get Fig. 28.

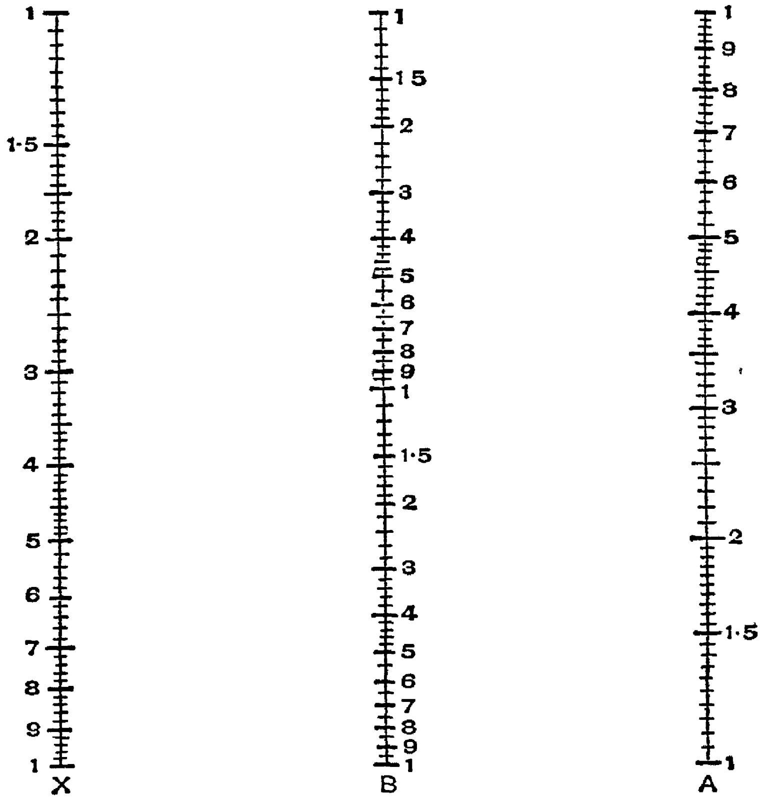


FIG. 28

The geometrical fact that for collinear points

twice distance $B =$ *distance* $A +$ *distance* X

(all these distances being measured from the bottom ends of the scales), now becomes

$$2\left(\tfrac{1}{2}-\tfrac{1}{2}\ \textit{log graduation } B\right)$$
$$= \textit{log graduation } A + (1 - \textit{log graduation } X),$$

so that

$$\textit{log graduation } X = \textit{log graduation } A + \textit{log graduation } B,$$

i.e. $X = AB$, or $B = X/A$, or $A = X/B$.

As in the use of the slide rule, so in our nomograms, we can operate with numbers of any size. (We can consider the factors in a multiplication or the dividend and divisor in a division as all being positive.) Thus to multiply 13×62, we use graduations 1·3 and 6·2 in A, B. The same graduations will be used for $1{\cdot}3 \times {\cdot}62$, or 130×620, or $\cdot 0013 \times 6200$, or $\cdot 013 \times \cdot 062$, or any other product in which the factors consist of these *significant* numbers 13 and 62. To divide 13 by 62 or any other factors having these significant numbers we use 1·3 and 6·2 in X and B (or X and A). In each case the position of the decimal point is decided by a rough check.

31. Generalised Multiplication and Division.

We can at once proceed to consider the general case of any number of multiplications and divisions of numbers raised to any given powers. Thus, suppose we have to work out

$$X = SA^l B^m C^n \ldots / F^p G^q \ldots,$$

where $l, m, n, \ldots, p, q, \ldots$ are positive (including fractional) given indices and $A, B, C, \ldots, F, G, \ldots$ are quantities whose values are at our disposal, whilst S is a constant factor. By taking logarithms, we get

$$\log X = l \log A + m \log B + n \log C \ldots$$
$$- p \log F - q \log G \ldots + \log S.$$

Let us write

$$\log A \equiv a, \ \log B \equiv b, \ \ldots, \ \log S \equiv s, \ \log X \equiv x.$$

Then the expression becomes

$$x = la + mb + nc \ldots - pf - qg \ldots + s.$$

Thus we have reduced the operation to one of *addition* and *subtraction* of a number of terms which are in fact the logarithms of the numbers we have to operate with. If then, we use scales in which the graduations give the values la, mb, etc., we have merely to make a nomogram for addition and subtraction. Now convert these scales into logarithmic scales, *i.e.* the graduation a is marked not a but A, the graduation b is marked B, and so on, and we have the nomogram for X in terms of A, B, C, etc. (If S is less than 1, s is negative.) Hence we have

Rule V. To construct a nomogram for

$$X = SA^{l}B^{m}C^{n}\ldots/F^{p}G^{q}\ldots$$

first construct a preliminary nomogram for

$$x = a + b + c \ldots - f - g \ldots .$$

Decide upon the length of a 0 to 1 interval for each scale according to the instructions given in Rule III., Chap. II, § 26, for the nomogram

$$la + mb + nc \ldots - pf - qg \ldots + s,$$

where s is log S, giving the zero of one of the scales so as to take account of s. Then convert each 0 to 1 interval into a 1 to 10 logarithmic scale, and use the resulting nomogram as giving X in terms of A, B, C ... .

We shall apply the method of this Chapter to some examples.

32. Nomogram for $X = AB^2$.

We first make a preliminary nomogram for

$$a + b,$$

i.e. we use the alternative nomogram for addition.

We convert this into a nomogram for

$$a+2b,$$

by doubling the b unit in the nomogram first obtained. We then convert each 0 to 1 interval into a 1 to 10 logarithmic scale. We do not put in zeros after the

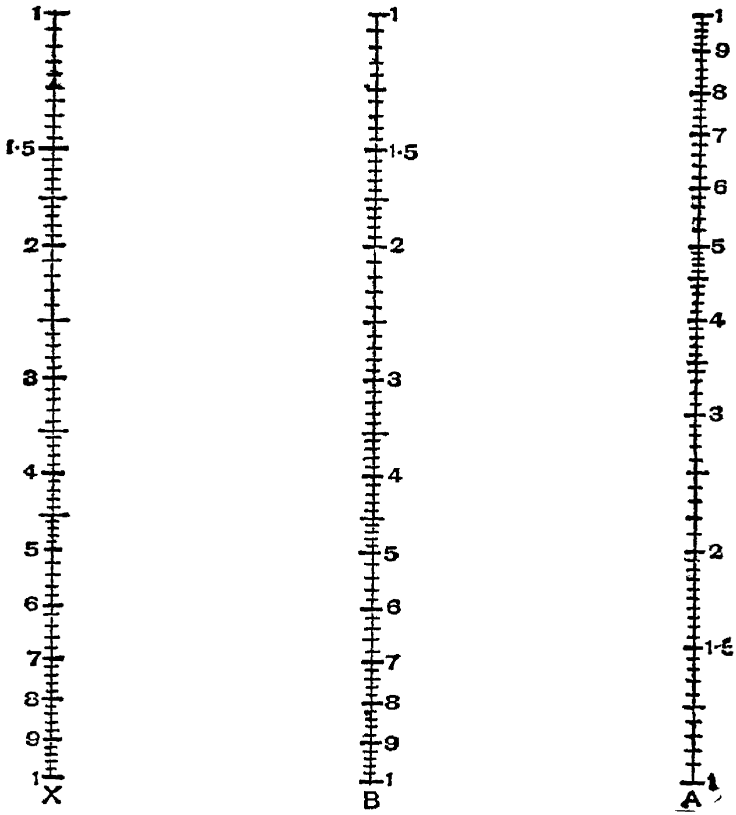


FIG. 29.

significant figure 1 in the end graduations : as already explained the exact meaning of the numerical interval represented by a 1 to 10 logarithmic interval can be considered as containing a power of 10, which is at our choice (Fig. 29). Thus

$$25 \times 3{\cdot}3^2 = 272.$$

33. Nomogram for $R = 0\ 00125SV^2$ (Resistance to a Parachute).

This formula represents the resistance in pounds exerted on a parachute of area S square feet, moving

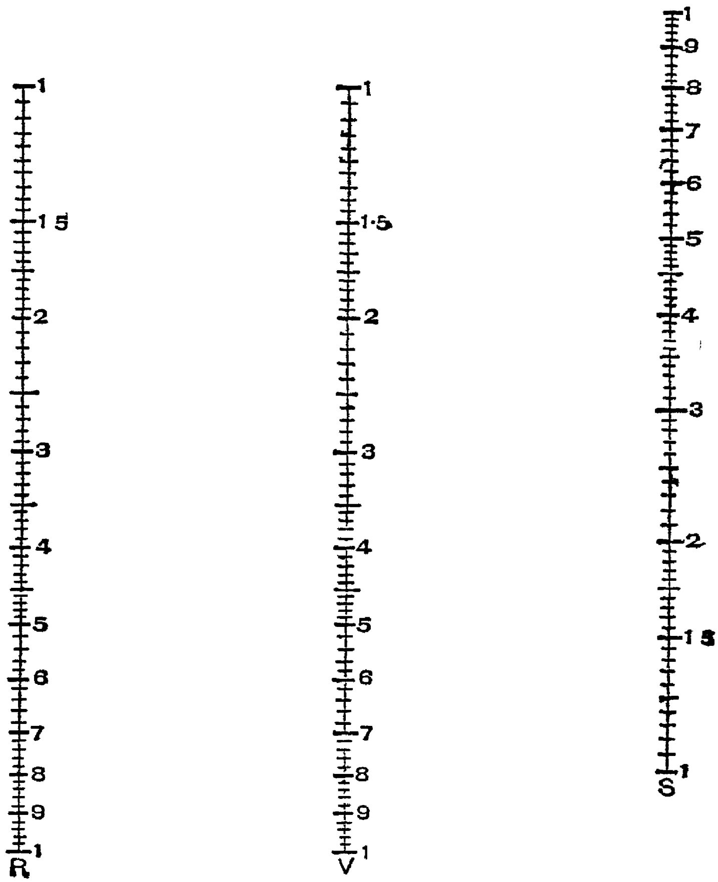


Fig 30

with velocity V feet per second through ordinary air (*Aviation Pocket Book*, 1918, p. 48). Writing it as

$$R = SV^2/800,$$

we first make a nomogram for

$$s + v,$$

where s is $\log S$, v is $\log V$, using the alternative

nomogram for addition. We then double the unit of v. Since $\log \frac{1}{800} = \bar{3}\cdot094$,
we take the zero for s to begin at the point ·094 on the s scale. We then convert into logarithmic scales (Fig. 30).

34. Nomogram for X=ABC.

We simply take the nomogram for

$$x=a+b+c,$$

given in Fig. 20, and convert each 0 to 1 interval into a 1 to 10 logarithmic scale (Fig. 31).

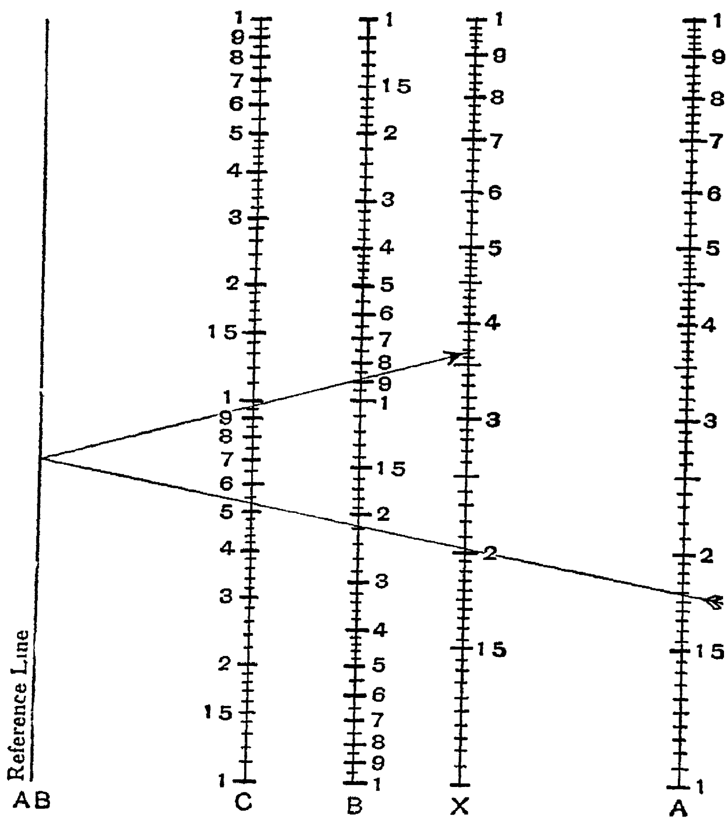


FIG 31

35. Nomogram for $R = 0.0194WSV^2$ (Air Pressure on a Plate).

In the introduction we mentioned the formula for the pressure R in pounds on a plate of area S square

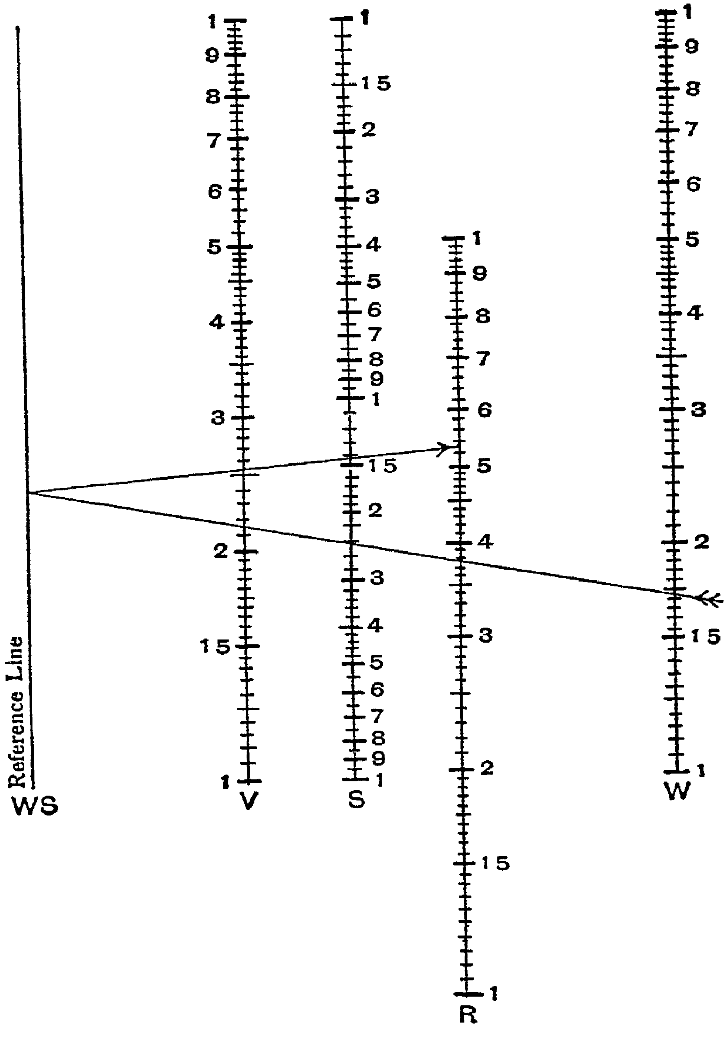


FIG 32

feet, past which air, weighing W pounds per cubic foot, is moving normally with a relative velocity of V

feet per second. The nomogram is given in Fig. 3. We can now show how the nomogram for this formula is constructed.

The preliminary nomogram is obtained by considering the product of three quantities, *i.e.* we start off with

$$r = w + s + v,$$

using Fig. 20. We double the unit of scale v. We have

$$\log 0{\cdot}0194 = \bar{2}{\cdot}288,$$

so that we put the zero of the r scale, at the point $-0{\cdot}288$. We then convert into logarithmic scales. (Fig. 32, which is the same as Fig. 3.)

In the practical use of this nomogram we must remember that W is really a small quantity, its value for air near the earth's surface being about 0·081. Thus, it may be well to say that W is graduated in hundredths of a pound per cubic foot and the decimal point in the answer has to be adjusted to get the correct result.

36. Nomogram for $W = 675BD^2/L$ (Breaking Load of Ash Beam).

The breaking load W (in pounds) for a beam of ash of length L feet, and rectangular cross section of sides B inches (breadth) and D inches (depth), which is supported at the ends and loaded at its centre, is given by the above formula. To construct a nomogram we first consider the preliminary nomogram

$$w = b + d - l$$

by means of the method of Fig. 22. We double the unit of d. We then find

$$\log 675 = 2{\cdot}83,$$

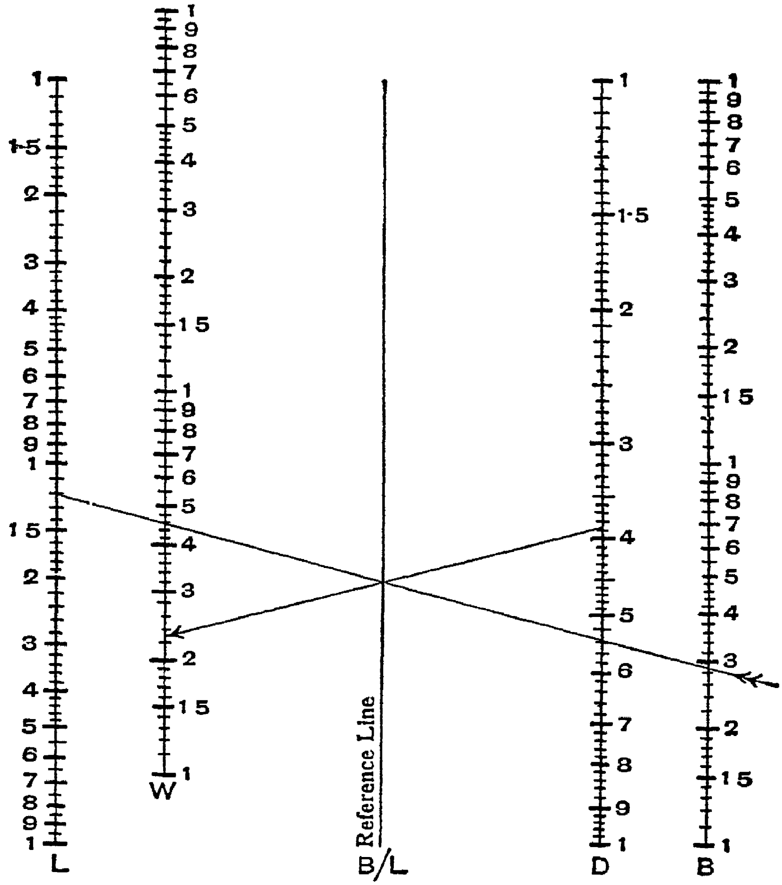


FIG 33

and we make the w scale have its zero at the point 0·17. Now we convert into logarithmic scales (Fig. 33).

37. Nomogram for $C = w/nd^2$ (Ballistic Constant)

This nomogram is given in Fig. 34, and its analysis is left as an exercise to the reader.

The quantity C is an important number used in gunnery, w is the weight of a projectile in lbs., d its diameter in inches, and n is a coefficient of reduction.

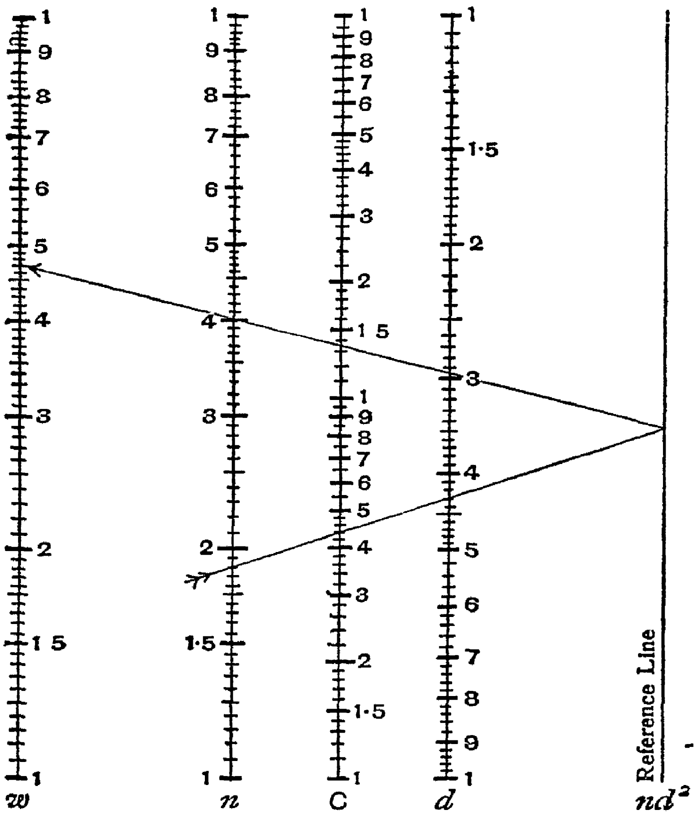


Fig 34.

38. Nomogram for $A = \frac{32W}{CDV^2}$ (Wing Area of Aeroplane).

If the weight of an aeroplane is W pounds and it is designed to fly with velocity V miles an hour horizontally in air weighing D pounds per cubic foot, then the area of wing surface required is A square feet, given by the above formula in which C is the "lift-coefficient" of the type of wing used, at the given angle of incidence (or attack). The preliminary nomogram is for

$$w-c-d-v,$$

which we construct according to rule III. The unit

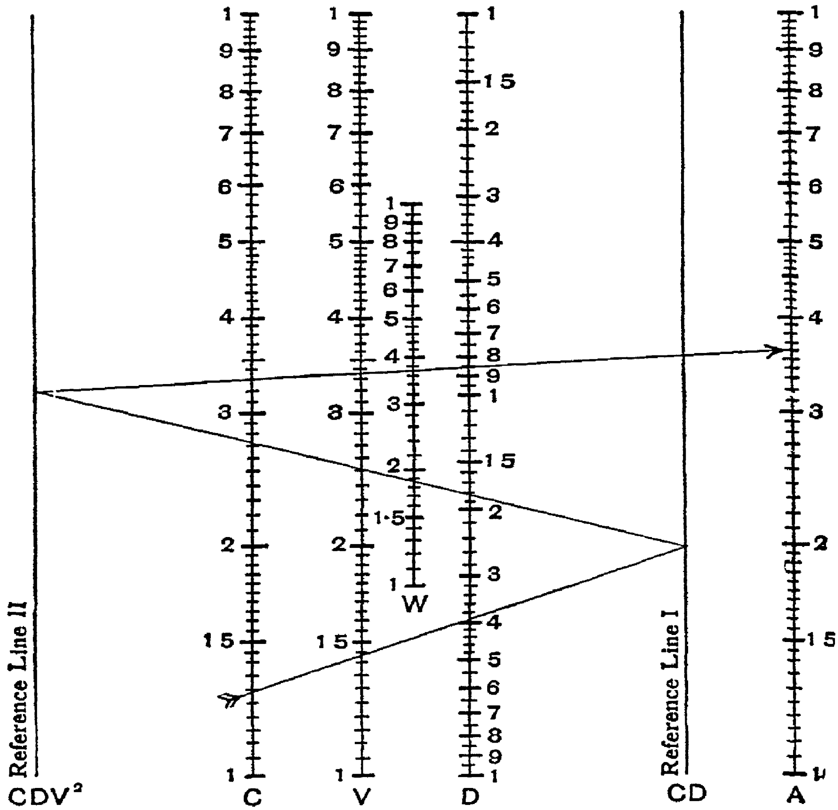


Fig 35

of the v scale is doubled and we graduate provisionally. Also, since

$$\log 32 = 1{\cdot}505,$$

we push up the zero of w to be at the point 1·505 as provisionally graduated. We then convert the 0 to 1 intervals into 1 to 10 logarithmic scales (Fig. 35).

39. Nomogram for $M = 270 \dfrac{WL^3}{EBD^3}$ (Maximum Deflexion of a Beam).

If a uniform beam of weight *W* pounds is supported

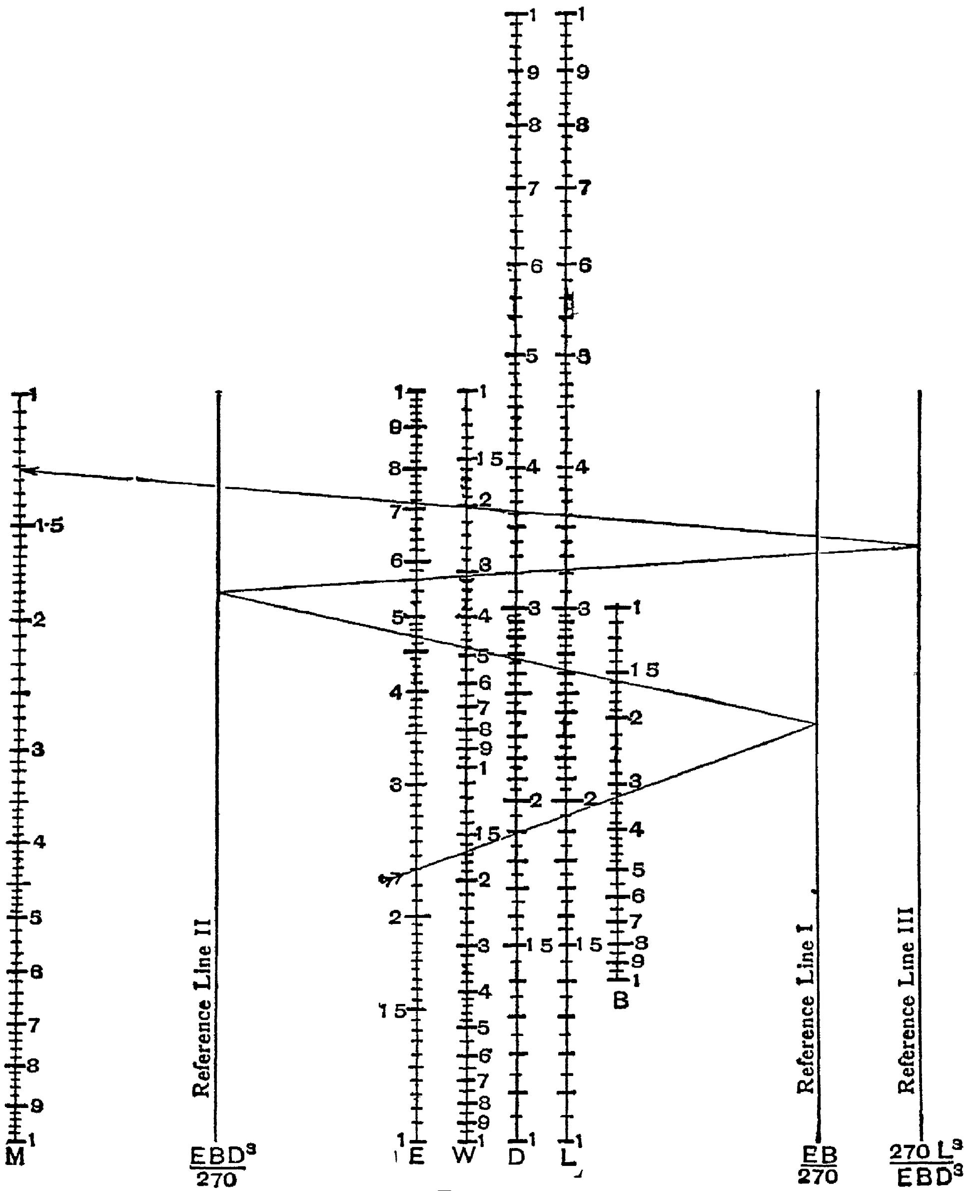


Fig 36.

horizontally at its ends, L being its length in feet, B, D the horizontal and vertical dimensions of its cross section in inches, and E the coefficient of elasticity in pounds per square inch, then the greatest deflexion M (which is at the middle point) is given by the above formula, in inches. The same formula holds for W and E both in tons (or any other unit).

The nomogram is given in Fig. 36, and the analysis of its construction is left as an exercise to the student.

EXAMPLES III.

1. Draw a logarithmic scale with a 25 cm. unit and put in all the useful subdivisions.

Do the same with a 15 cm. unit.

2. Draw a logarithmic scale with an 8 cm. unit.

3. Construct nomograms for

(i) $X=A^3B^2$; (ii) $X=AB^4$; (iii) $X=A/B^2$;
(iv) $X=A^2/B$; (v) $X=A^{\frac{3}{2}}B$; (vi) $X=A/B^{\frac{1}{2}}$.

4. Construct nomograms for

(i) $X=3A^2/B^3$; (ii) $4X=7A^{\frac{1}{2}}B^{-\frac{1}{2}}$;
(iii) $X=\frac{1}{5}A^{-1}B^{-2}$; (iv) $s=\frac{1}{2}gt^2$;
(v) $h=v^2/2g$; (vi) $E=\frac{1}{64}WV^2$;
(vii) $V=\frac{\pi}{3}a^2h$; (viii) $V=\frac{4}{3}\pi\rho a^3/r$;
(ix) $F=4\pi a^2\sigma/r^2$; (x) $\text{H.P.}=\frac{RV}{550}$;
(xi) $V=\frac{33000\times\text{H.P.}}{W}$; (xii) $t=2\pi\sqrt{\frac{l}{g}}$.

5. Construct a nomogram for $d=2{\cdot}88\sqrt[3]{\frac{\text{H.P.}}{N}}$, where d is the diameter of a shaft in inches, H.P. is the horse power transmitted, and N is the number of revolutions per minute.

6. If W is the weight of a body in lbs., r its distance from an axis of rotation in feet, N the number of revolutions per minute,

then the centrifugal force in lbs. is $0{\cdot}000343\,WrN^2$. Construct a nomogram for this.

7. The radius R in feet of a metallic sphere weighing W lbs., made of specific gravity ρ, is given by $\frac{4}{3}\pi\rho R^3 = W/62\frac{1}{2}$ Construct a nomogram for R in terms of ρ and W.

8. The time in seconds of oscillation of a pendulum is $2\pi\sqrt{\dfrac{k^2}{hg}}$ where k is the radius of gyration, and h is the distance of the centre of gravity from the axis of rotation Construct a nomogram using centimetres for k, h, g. (Take $g = 981$ cm./sec^2.)

9. Construct a nomogram for $T^2 = \dfrac{4\pi a^3}{\mu}$.

10. Construct a nomogram for $3WND^3/1000d^4$.

CHAPTER IV

NOMOGRAMS WITH TWO PARALLEL SCALES
QUADRATIC EQUATIONS, ETC

40. The Quadratic Equation.

It is a very easy matter to construct a nomogram for the type of quadratic equation

$$x^2+ax+b=0,$$

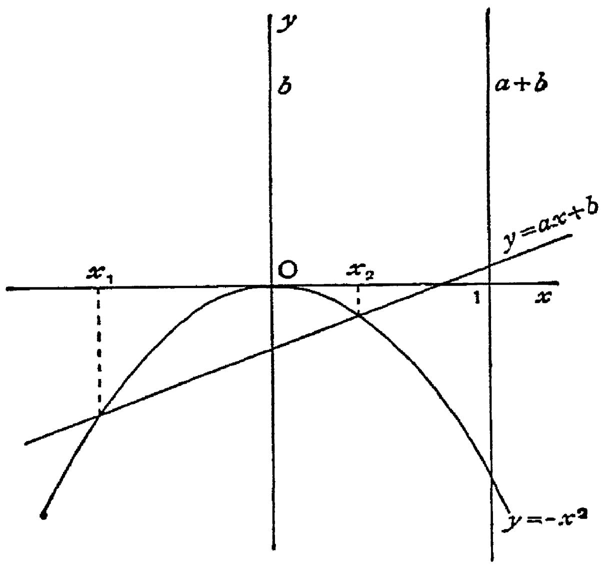


FIG 37

for all values of a and b giving real solutions. For if we plot the parabola

$$y=-x^2,$$

and the straight line

$$y = ax + b,$$

it is clear that the x *coordinate* of a point of intersection of the parabola and the line *is a solution of our equation.* We only need some easy mode of drawing the line $y = ax + b$ for all values of a, b, since the parabola *can be drawn once for all.*

To obtain the line we note that

$$\text{when } x = 0, \quad y = b,$$

and

$$\text{when } x = 1, \quad y = a + b.$$

Thus, we take the point b on the y axis, and the point $a + b$ on the line $x = 1$, *i.e.* parallel to and at unit distance from the y axis (Fig. 37).

41. Symmetrical Nomogram.

The nomogram in § 40 is lacking in symmetry. To restore the symmetry we note that

$$\text{when } x = -1, \quad y = b - a = -(a - b).$$

Hence we take two parallel lines at distance two x units apart, and graduate them uniformly in opposite directions with some convenient y unit (which may be different from the one already used, Fig. 38). We then plot the parabola $y = -x^2$ and graduate it so that at any point is given the x coordinate (in terms of the x unit). Then if we take a point $a + b$ on one scale and $a - b$ on the other, the join cuts the parabola at a point whose graduation is a solution of the quadratic equation. In Fig. 38 the x unit is 10 times the y unit.

The figure is drawn so as to give solutions between $+2$ and -2.

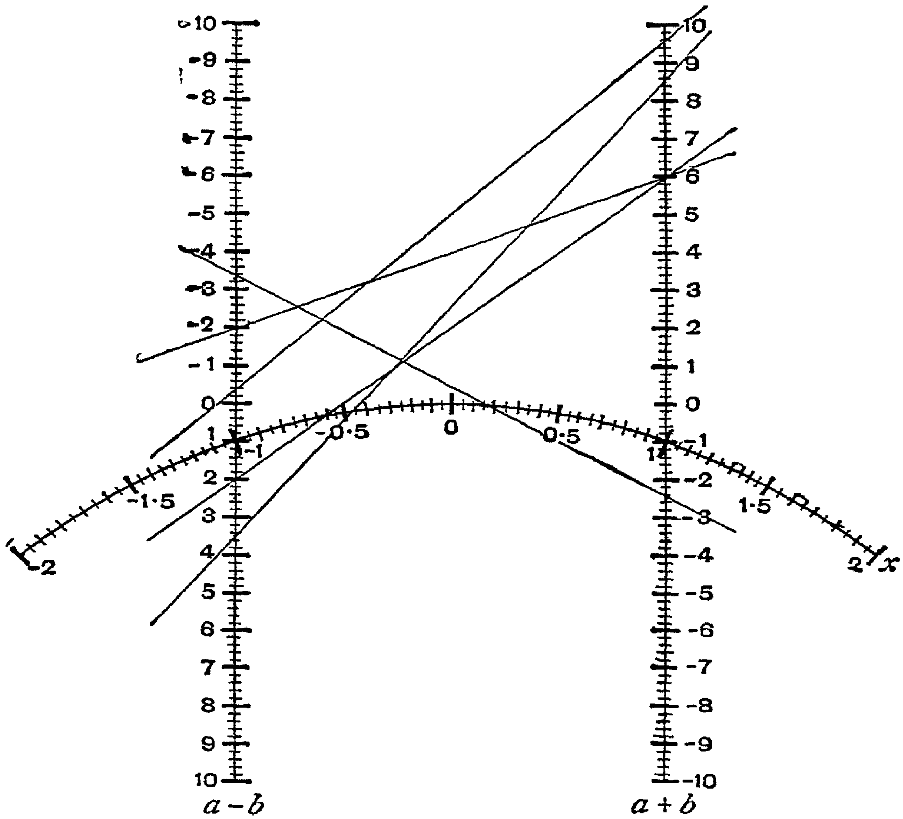


FIG. 38

42. Use of the Nomogram.

In practice only one root need be found. This depends upon the fact that if x_1, x_2 are the two solutions of the equation

$$x^2+ax+b=0,$$

then $$x_1+x_2=-a, \quad x_1x_2=b.$$

Thus if we have found x_1 from the nomogram, x_2 is given as

$$x_2=-a-x_1.$$

A good check is then provided by the remaining relation $x_1x_2=b$.

Thus, if *one* of the solutions of the quadratic equation lies between +2 and −2, the equation can be solved completely. *E.g.* take

$$x^2+4x+2=0,$$

so that $a=4$, $b=2$. We take the points 6 on the $a+b$ scale and 2 on the $a-b$ scale. The join cuts the parabola at the point whose x coordinate, and therefore graduation, is −0·585. Hence

$$x_1=-0{\cdot}585,$$
$$x_2=-4+0{\cdot}585=-3{\cdot}415,$$

and it is found that $(-0{\cdot}585)\times(-3{\cdot}415)$ is nearly 2.

It may happen, however, that the numbers $a+b$, $a-b$, are too big to be included in the nomogram, as *e.g.* in

$$x^2+12x+10=0,$$

in which $a=12$, $b=10$. We can then reduce a and b by means of the following device.

Put
$$x=2x',$$

and the equation becomes

$$4x'^2+24x'+10=0,$$

i.e.
$$x'^2+6x'+2{\cdot}5=0,$$

in which $a=6$, $b=2{\cdot}5$. We then find

$$x_1'=-0{\cdot}45,$$

so that
$$x_1=-0{\cdot}9,$$

and therefore
$$x_2=-12+0{\cdot}9=-11{\cdot}1.$$

The product x_1x_2 is sufficiently close to 10.

Again, if we have the equation

$$x^2-29x+53=0,$$

we put
$$x=10x',$$

so that
$$x'^2-2{\cdot}9x'+0{\cdot}53=0,$$

giving $a=-2{\cdot}9$, $b=0{\cdot}53$. We get from the nomogram

$$x_1' = 0{\cdot}195,$$

so that

$$x_1 = 1{\cdot}95,$$

and therefore

$$x_2 = 27{\cdot}05.$$

The product check is readily verified.

In any such case one very soon finds a convenient change (or transformation) from x into x'.

43. Cases of Failure.

It may happen that the line joining the points $a+b$, $a-b$ does not cut the piece of the parabola given in the nomogram. Thus,

$$x^2+2x+4=0$$

gives $a+b=6$, $a-b=-2$, and the join does not cut the curve. This may be the consequence of one of two causes. Either the line would never cut the parabola at all even if the parabola were plotted to infinity. Or the line does cut the parabola but so far away that neither of the two intersections is on the nomogram as actually drawn. In the former case the quadratic equation given has no real solution. In the latter case real solutions do exist, and they should be found. It is often possible to judge from the look of the diagram whether we have the first or the second case. Thus, $a+b=6$, $a-b=-2$ gives a line that evidently does not cut the parabola at all. But the equation

$$x^2+4\ 6x+5=0$$

gives $a+b=9{\cdot}6$, $a-b=-0\ 4$, and the line through these points may or may not cut the parabola—the diagram is not convincing either way. To make quite sure in such a case, we again apply the device already given. Thus, put

$$x = 2x',$$

we get $$x'^2 + 2{\cdot}3x + 1{\cdot}25 = 0,$$

so that $a+b=3{\cdot}55$, $a-b=1{\cdot}05$. We find

$$x_1' = -0{\cdot}84,$$

so that $$x_1 = -1{\cdot}68, \quad x_2 = -4{\cdot}6 - (-1{\cdot}68) = -2{\cdot}92,$$

the product being sufficiently close to 5.

44. Cubic Equations

The cubic equation is one involving the third power of the unknown : such an equation as

$$x^3 + 3x^2 - 2x + 5 = 0,$$

or $$2x^3 + x^2 - 1 = 0,$$

or $$x^3 - 4x + 7 = 0,$$

is a cubic equation in x. It can be easily proved that by means of a simple device, such an equation can be made to contain only three terms, *i.e.* the cubic term and two of the other three terms. In particular, it is useful to make the equation contain only the cubic term, the first order term (*i e.* x) and the term with no x in it: in other words, the square term (x^2) can be eliminated. The equation is then said to be of Cardan's form.

Thus, take the equation

$$x^3 + 3x^2 - 2x + 5 = 0.$$

Put $$x + 1 = x' ;$$

the equation is

$$(x'-1)^3 + 3(x'-1)^2 - 2(x'-1) + 5 = 0,$$

i.e. after a little algebra,

$$x'^3-5x'+9=0,$$

in which the square term (x'^2) is missing.

The equation $\quad 2x^3+x^2-1=0,$

can be written $\quad x^3+\frac{1}{2}x^2-\frac{1}{2}=0.$

Put $\quad x+\frac{1}{6}=x',$

and we get $\quad (x'-\frac{1}{6})^3+\frac{1}{2}(x'-\frac{1}{6})^2-\frac{1}{2}=0,$

i.e. $\quad x'^3-\frac{1}{12}x'-\frac{53}{108}=0.$

The last equation above is already of the required form.

To remove the square term when it exists, we first make the coefficient of x^3 unity. Thus, if the equation is

$$ax^3+bx^2+cx+d=0,$$

we divide by a and get

$$x^3+\frac{b}{a}x^2+\frac{c}{a}x+\frac{d}{a}=0.$$

We then put $\quad x+\lambda=x',$

so that we get by substitution

$$(x'-\lambda)^3+\frac{b}{a}(x'-\lambda)^2+\frac{c}{a}(x'-\lambda)+\frac{d}{a}=0,$$

i.e. $$x'^3+\left(\frac{b}{a}-3\lambda\right)x'^2+\left(\frac{c}{a}-2\lambda\frac{b}{a}+3\lambda^2\right)x' +\left(\frac{d}{a}-\lambda\frac{c}{a}+\lambda^2\frac{b}{a}-\lambda^3\right)=0.$$

Hence the term x'^2 disappears if we have

$$\lambda=\frac{1}{3}\frac{b}{a}.$$

We thus write $\quad x+\frac{1}{3}\frac{b}{a}=x',$

and the equation becomes

$$\left(x' - \frac{1}{3}\frac{b}{a}\right)^3 + \frac{b}{a}\left(x' - \frac{1}{3}\frac{b}{a}\right)^2 + \frac{c}{a}\left(x' - \frac{1}{3}\frac{b}{a}\right) + \frac{d}{a} = 0,$$

and it is at once seen that the equation in x' does not contain the square term.

This property of the cubic equation is particularly valuable in the construction of a nomogram, because

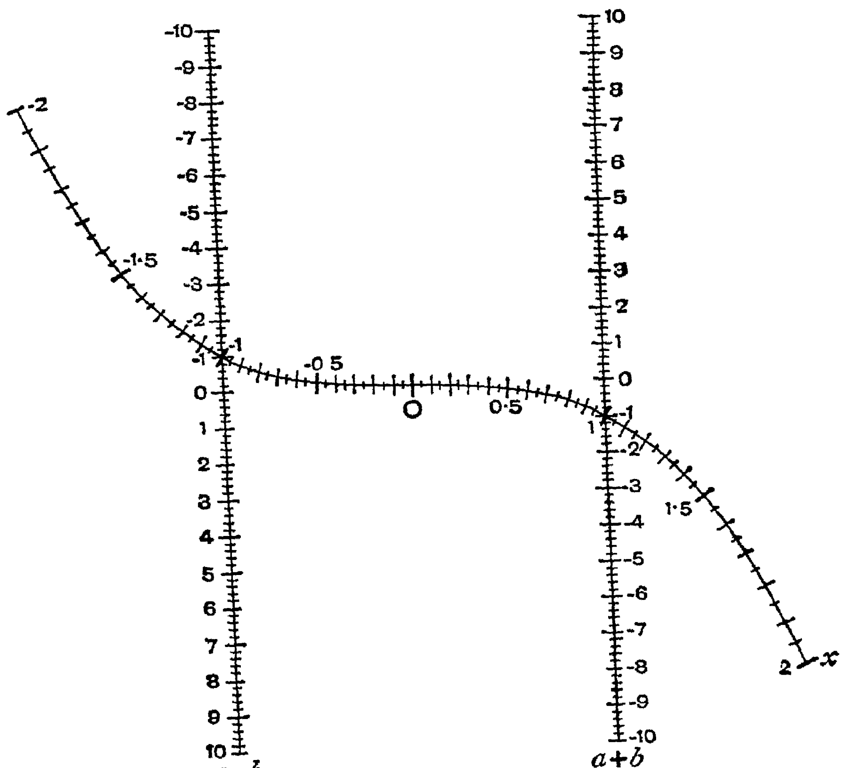


FIG. 39.

a method exactly similar to that of § 40 can be applied to the equation $x^3 + ax + b = 0$.

We plot $y = -x^3$;

the line $y=ax+b$ cuts this curve at points whose x coordinates are the solutions of the cubic. It is clear that the line $y=ax+b$ can be obtained for different values of a, b in exactly the same way as in the case of the quadratic equation. The construction of the nomogram thus becomes merely the substitution of the curve $y=-x^3$ (cubical parabola) for the curve $y=-x^2$ in Fig. 38.

This has been done in Fig. 39, which gives solutions ranging between $+2$ and -2. Cases in which $a+b$ or $a-b$ is too big, or in which the join does not cut the curve as given in the nomogram, can be dealt with exactly as explained for the quadratic.

45. Extension.

But an equation of a degree higher than the third cannot always be treated in such a way. The reason is that such an equation cannot always be made to consist of only three terms. It may by accident be of this form, or be reducible to this form, but in general this will not be the case.

Thus, if in the equation

$$x^4+4x^3+6x^2+7x+5=0,$$

we write $$x+1=x',$$

we get $$x'^4+3x'+1=0.$$

But the equation

$$x^4+4x^3+4x^2+7x+5=0,$$

cannot be reduced quite so much. We can remove the term in x^3 by using the same device, but the term in x^2 remains, as the student will readily verify.

The same holds of equations of the fifth and higher degrees: in each case we can remove the term after the highest, but we must not expect to be able to do more in general.

But it may happen that an equation of the fourth degree takes the form

$$x^4+ax+b=0,$$

or an equation of the fifth degree takes the form

$$x^5+ax+b=0,$$

or an equation of the nth degree takes the form

$$x^n+ax+b=0.$$

In such a case we can once more use the method of § 40.

If we have an equation

$$x^n+ax+b=0,$$

we plot $$y=-x^n,$$

instead of $y=-x^2$ in Fig. 38. It is not necessary for n to be an integer and positive; it can be any number.

Finally, if we have an equation

$$x^n+ax^m+b=0,$$

we put $$x=(x')^{\frac{1}{m}},$$

so that $$x^n=(x')^{\frac{n}{m}},$$

and we get $$(x')^{\frac{n}{m}}+ax'+b=0.$$

We solve this by means of a nomogram consisting of Fig 38, with the curve $y=-(x')^{\frac{n}{m}}$ substituted for $y=-x^2$. Then using $a+b$, $a-b$ as for the quadratic, we find a value or values of x' satisfying the derived equation. The corresponding values of x are then readily found (see Ch. VII. § 69).

Thus the equation

$$x^3 + ax^{\frac{5}{7}} + b = 0$$

is solved by means of putting

$$x = (x')^{\frac{7}{5}},$$

so that $$(x')^{\frac{21}{5}} + ax' + b = 0,$$

the curve in the nomogram being

$$y = -(x')^{\frac{21}{5}}.$$

46. The nomograms just obtained are, however, not convenient for practical use because of the necessity of calculating $a+b$, $a-b$. It is desirable to construct a nomogram in which the numbers a, b are used *directly*, *i.e.* in which we take a point a on one scale, and a point b on another scale, and let the join cut a curve graduated in such a way as to give the required solution. This is the case in the nomograms of the first three chapters.

47. D'Ocagne's Nomogram for $x^2 + ax + b = 0$.

Before proceeding to a discussion of the method of constructing these more convenient nomograms, we shall show that such are possible by giving the nomogram for the quadratic equation made by d'Ocagne. We shall show that d'Ocagne's method is correct: later on, in Chapter V., we shall show how this and other nomograms can be discovered.

Take a pair of axes $O\xi$, $O\eta$ at right angles to one another and plot the curve (hyperbola),

$$\eta = -\frac{\xi^2}{\xi+1}$$

between the values $\xi = -1$ and $\xi = 0$. In Fig. 40, we have made the ξ unit ten times the η unit. Draw

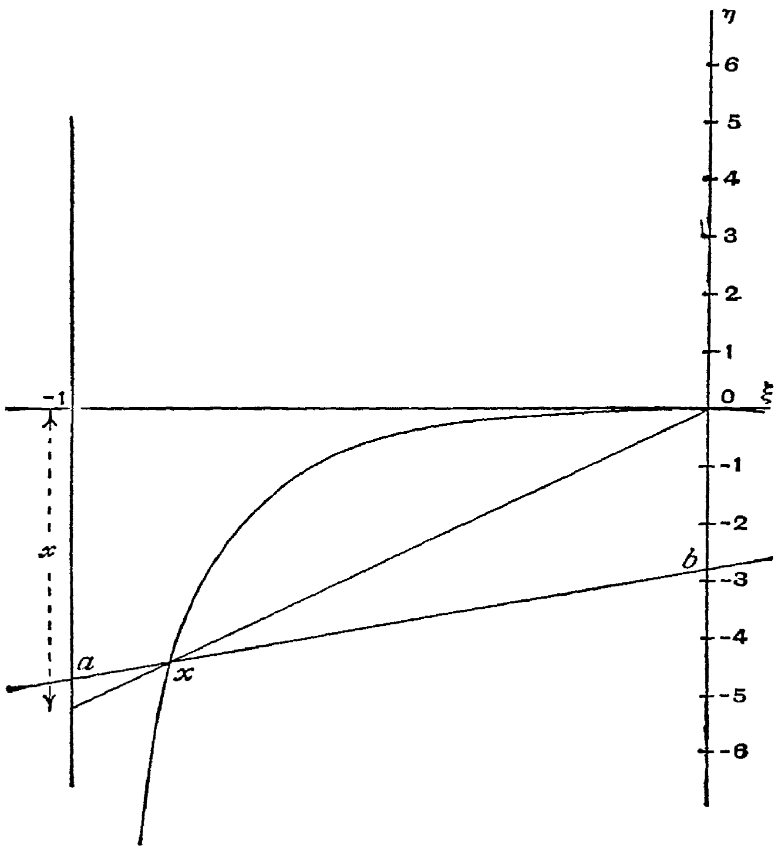


FIG 40.

the line $\xi = -1$, *i.e.* parallel to the η axis and at distance one ξ unit to the left. This line is an asymptote of the hyperbola as plotted.

If we join the origin to any point on the asymptote at distance x (η units) *below* the ξ axis, then this line cuts the hyperbola at a point whose ξ, η coordinates obey the relation

$$\frac{\eta}{\xi}=x.$$

But
$$\frac{\eta}{\xi}=-\frac{\xi}{\xi+1}$$

for all points on the hyperbola Hence we have

$$\eta=\xi x \quad \text{and} \quad x=-\frac{\xi}{\xi+1}.$$

Let us call this point the x point on the hyperbola.

Now let any line through this x point cut the η axis at the point b, and the asymptote at the point a, so that b is the number of η units in the distance of the first point from the origin, and a is the number of η units in the distance of the second point from the ξ axis. (In Fig 40 both are negative, the first is $b=-2{\cdot}8$, the second is $a=-4{\cdot}7$.) The equation of this line is

$$\eta=b+(b-a)\xi.$$

But the line also passes through the x point on the hyperbola. Further

$$\eta=\xi x$$

as already proved We therefore get

$$\xi x=b+(b-a)\xi.$$

The equation
$$x=-\frac{\xi}{\xi+1},$$

gives
$$\xi=-\frac{x}{x+1}.$$

Hence we have for x the equation

$$-\frac{x^2}{x+1}=b-(b-a)\frac{x}{x+1},$$

which reduces to $x^2+ax+b=0.$

Rule VI. To construct a nomogram for the equation

$$\mathbf{x^2+ax+b=0,}$$

take a pair of perpendicular axes Oξ, Oη; plot the hyperbola

$$\eta=-\frac{\xi^2}{\xi+1}$$

between $\xi=-1$ and $\xi=0$. Draw the asymptote $\xi=-1$. Graduate the η axis in η units and call it the b scale, graduate the asymptote with the same unit and call it the a scale. Join the origin successively to the negative graduations on the a scale and where any join cuts the hyperbola, put down the corresponding positive number: in this way put in all the useful graduations on the hyperbola Then a line joining the point a on the a scale to the point b on the b scale cuts the curve at the graduation x, which is a solution of the quadratic equation.

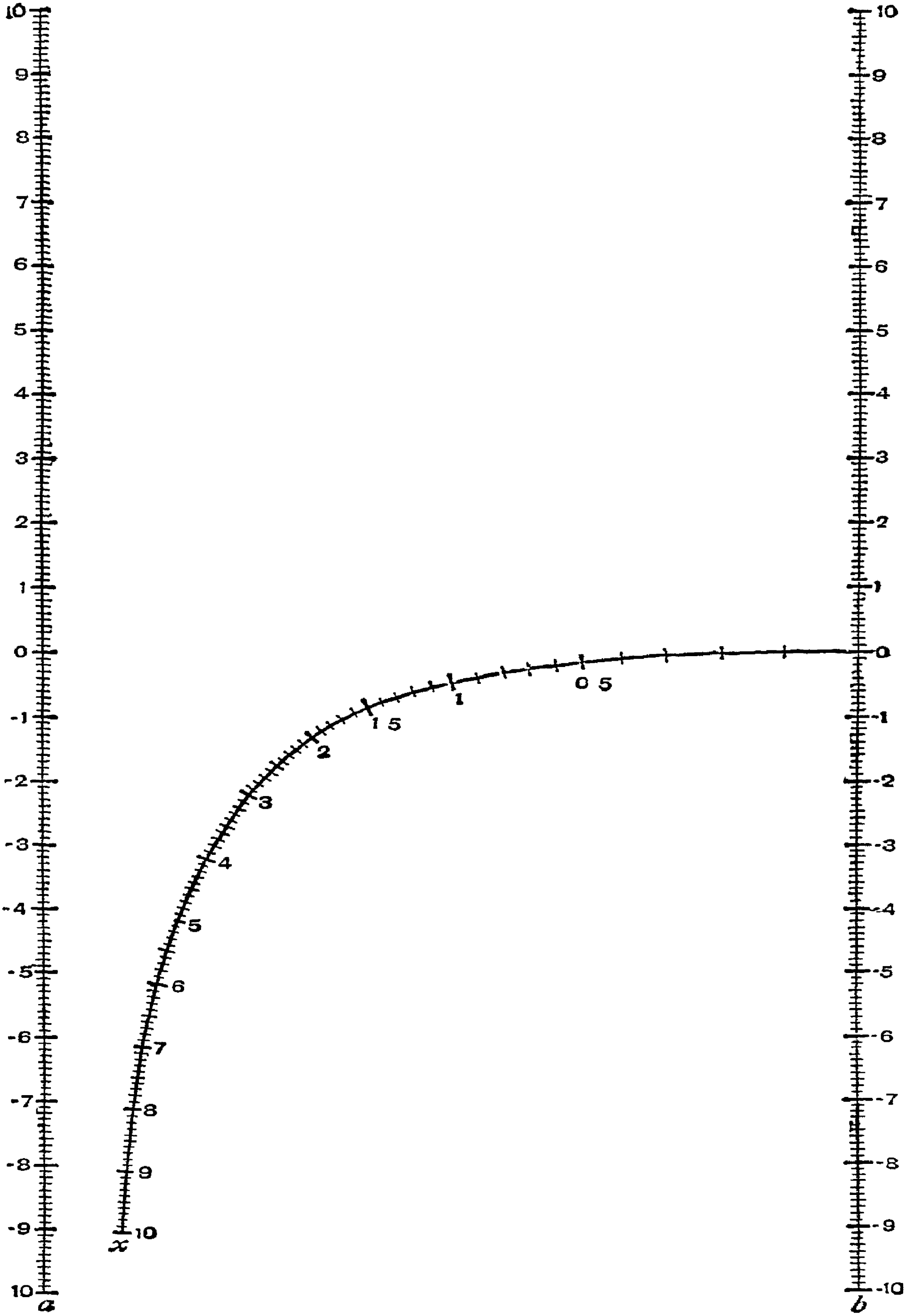
The nomogram as drawn in Fig. 41 gives only positive solutions ranging from 0 to 10. In fact, we need to consider only positive solutions For if *one* of the solutions of the quadratic is negative, we can find the positive solution x_1, and then find $x_2=-a-x_2$ as before (§ 42). If both solutions are negative, we put

$$x=-x',$$

and the equation becomes

$$x'^2-ax'+b=0,$$

in which both solutions are positive. Having solved



a
b
x
0 5
1
1 5
2
3
4
5
6
7
8
9
10

this equation we merely reverse the signs to get the solutions of the original equation.

For solutions above 10 we can use the method given in § 42.

48. Quadratic Equation with given Ranges of the Coefficients.

The nomogram as given in Fig. 41 is suitable for cases in which the values of a, b are in no way circumscribed. In practical cases these coefficients will lie between definite limits, and it is desirable to construct the nomogram in such a way as to take advantage of this circumstance. We shall illustrate the method by considering a case where we know *a priori* that in the equation

$$x^2+ax+b=0,$$

a is positive and never greater than 5, b negative and never less than -5.

Let us use a pair of axes $O\xi$, $O\eta$ inclined to one another. The general theory of "oblique axes" is of some difficulty. But here we shall only use them in a way that can be readily understood by anybody familiar with graphs and easy geometry of similar figures. We can plot a curve on "rhombussed" paper (Fig. 42) just as well as on squared paper. The only thing we need to find out is the equation of a straight line. It is at once seen that if BP is a line cutting the η axis at B, and PN, BN are parallel to the axes, then

$$NP/BN$$

is the same for all points on the straight line, say m. Hence

$$NP=m \cdot BN=m\xi,$$

where ξ is the abscissa of P, and

$$\eta = OB + NP = m\xi + b,$$

where η is the ordinate of P, and b is the distance OB, the intercept on the η axis. If the line cuts $\xi = -1$

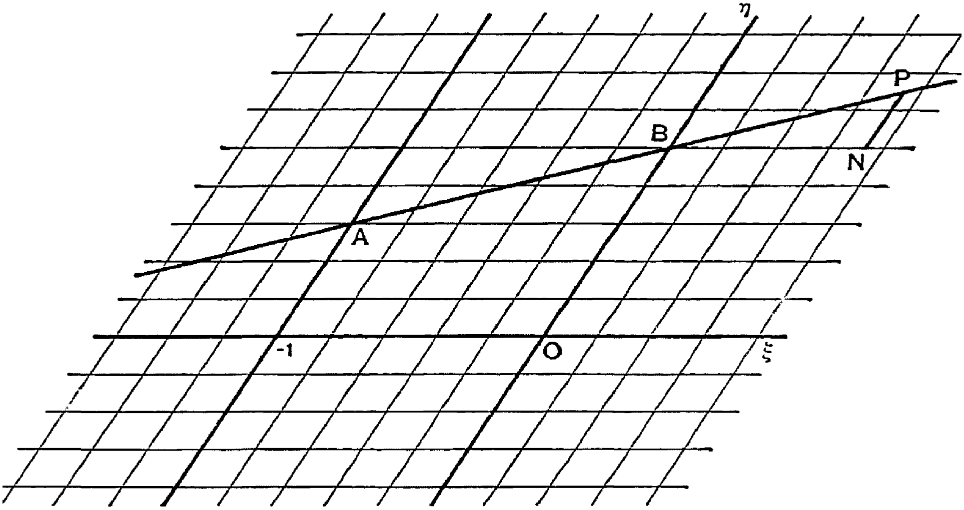


FIG. 42.

at A, then the ordinate of the point of intersection is $a = -m + b$. Thus a line which cuts the η axis and the line $\xi = -1$ at oblique distances a, b from the ξ axis has the equation

$$\eta = b + (b - a)\xi,$$

exactly as in the case of rectangular coordinates.

If then we plot the curve

$$\eta = -\frac{\xi^2}{\xi + 1},$$

and carry out all the process of the last article, we shall get once again a nomogram for the quadratic equation.

Now let the angle between the axes be the angle whose tangent is 2, and let the ξ unit be $5\sqrt{5}$ times the η unit. It will be found that not only is the part of the b scale between 0 and -5 equal and parallel to the part of the a scale between 0 and $+5$, but that these two lengths are opposite sides of a rectangle, so that the nomogram can be conveniently constructed as in Fig. 43.

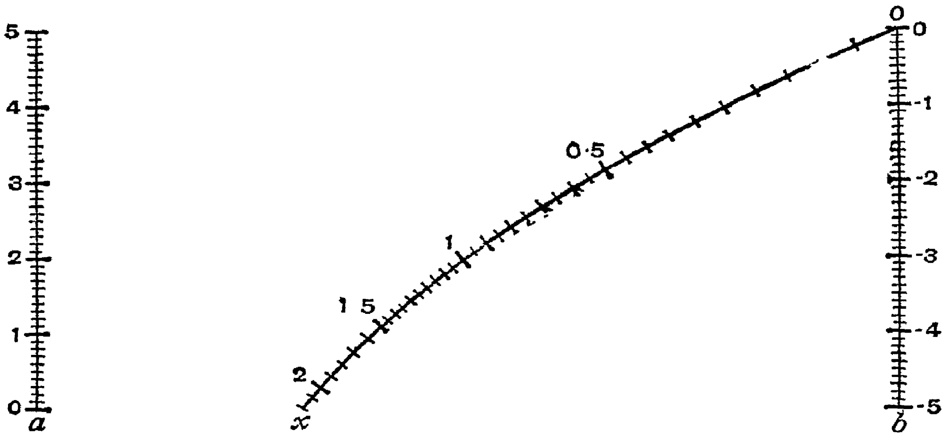


FIG 43

The nomogram thus made is more suitable for the given ranges of a, b than the general nomogram in Fig. 41. It has the additional great advantage that owing to the prescribed ranges the nomogram can be made bigger and therefore more accurate.

49. Automatic Method

Let us consider the case where a ranges up to $+1400$, and b down to -2500. To get the useful parts of the a, b scales in convenient relative positions we take the angle between the axes to be 45°, and choose the ξ unit to be 2500 times the η unit. Since, however, rhombussed paper is not a marketable article (especially as the angle between the axes useful in one case may be useless in another), it is desirable to be able to plot the curve in the nomogram without the use of oblique coordinates. Having once convinced ourselves that the nomogram *can* be constructed, we can evidently choose any method that gives us the result in the shortest time and with the least labour. Such a method is now to be given in detail for the case we are discussing.

In Fig. 44 we have the a, b scales placed in such a way that the join of their zeros cuts them at 45°. The a scale is graduated from 0 to 1400, the b scale from 0 to -2500, the unit being such that the distance between the zeros is 2500. (Of course any other distance could be used, but the one chosen gives a convenient figure.) To plot and graduate the x curve we can proceed as follows :

We know that any x graduation is a solution of the quadratic equation $x^2+ax+b=0$, whose coefficients a, b are the graduations at the points where any line through the point x cuts the a, b scales. If then we find *two* lines whose a, b intercepts give equations each of which has a certain solution x, the intersection of these lines must be the point x on the curve looked for. *We thus get both a point on the curve and the corresponding x graduation.*

Since b varies between 0 and -2500, it follows that the useful part of the required curve must lie between $x=0$, $x=50$. Let us then find the point on the curve for any value of x between these limits, say, $x=5$

We need *two* equations each of which has a solution

5, with a, b lying within the given limits. An obvious one is $x^2-25=0$. If we try to draw this line we see

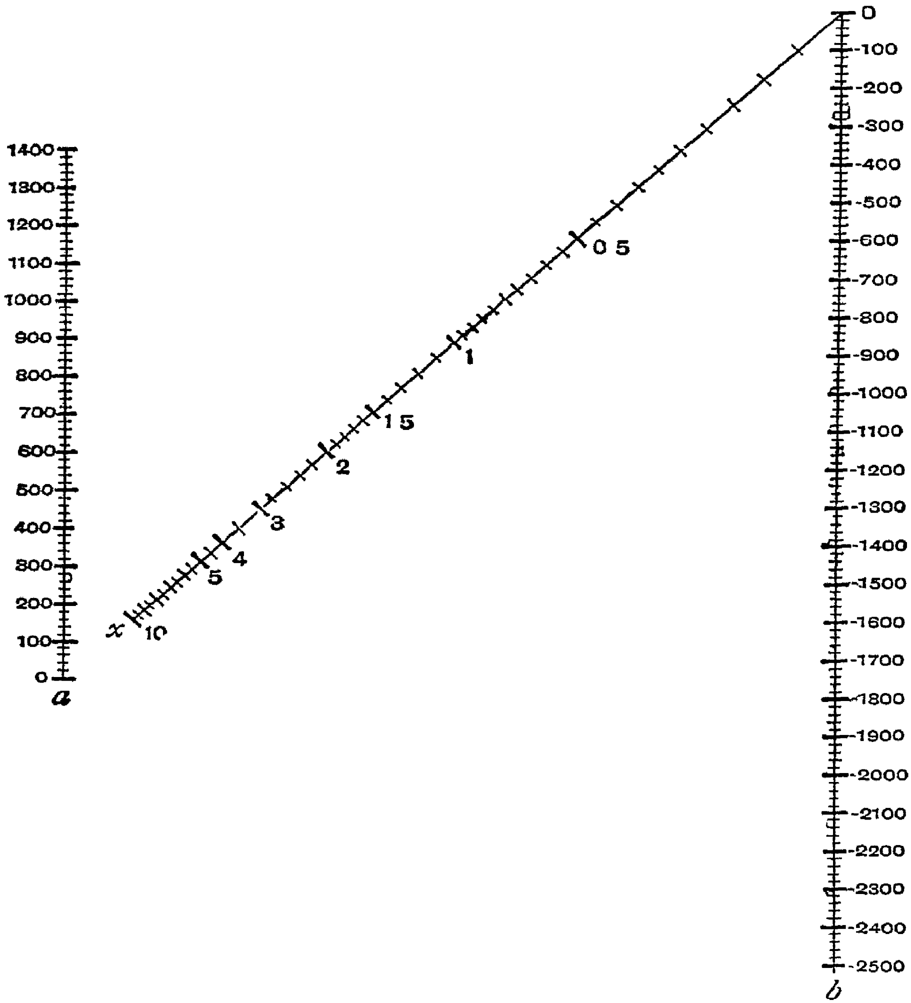


Fig. 44

that it is very nearly coincident with the line of zeros. To get another line as distinct as possible we use

$$(x-5)(x+500)=0,$$

i.e. $$a=495, \quad b=-2500.$$

Similarly for any other value of x.

In the particular case under discussion it happens that the values $x=0$ to $x=$about 10 give points practically on the line of zeros. This is, of course, an accident—but a useful accident. For it enables us to put in the graduations between 0 and 10 by just finding one additional line for any value of x—the line of zeros giving the required intersection point: it is, in fact, very approximately the x curve. Fig. 44 has been constructed in this way.

When x is considerably greater than 10, the curve deviates from the line of zeros, but never very much. In any case we do not really need these graduations as already explained (§ 42).

50. Quadratics with Widely Different Ranges of the Coefficients.

If the ranges of a, b are widely different it is clearly inconvenient to use the same unit in both the a, b scales. But by means of the automatic method introduced in the last article we can at once make a nomogram suitable for the given ranges.

Suppose then that a is always positive and less than 100, b negative and numerically less than 8000. The equation

$$x^2+ax+b=0,$$

can be written

$$x'^2+a'x'+b'=0,$$

where

$$x=100x', \quad a'=\frac{a}{100}, \quad b'=\frac{b}{10{,}000},$$

and the equation in x' has coefficients with more or less equal ranges. If then we make a nomogram for x' and multiply the x' graduations by 100, the a'

by 100, and the b' by 10,000, we shall have the nomogram for the given ranges.

Since a nomogram exists, we can proceed to construct it straightforwardly by the method of the last article.

In Fig. 45 we have made the a unit 100 times the b unit, the line of zeros being at 30° with the a, b scales,

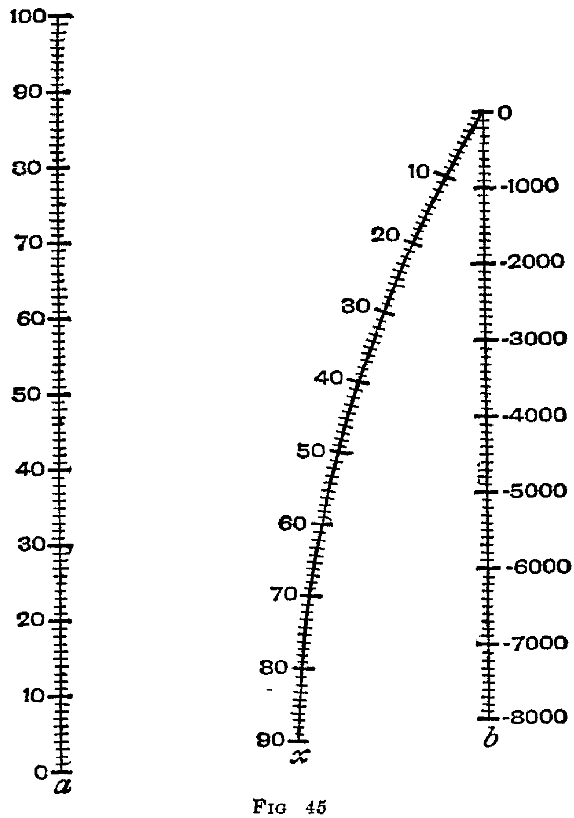


FIG 45

and the distance between the zeros 100 a units, so that the a, b scales form a conveniently shaped quadrilateral. The solution x is taken to range between 0 and 90. For an x graduation equal to 40, typical of graduations between 20 and 90, we use two lines given by

$$x^2 - 1600 = 0, \quad x^2 + 60x - 4000 = 0.$$

For graduations between 0 and 20 the former type of equation is not convenient. As a matter of fact this part of the curve lies near the line of zeros and can be readily drawn by freehand as a continuation of the x curve from 90 to 20. It can then be graduated by means of the second type of equation.

It happens that the x curve is graduated in what we may call an approximately regular manner. The subdivisions can thus be put in with great ease.

The reader will now be able to construct nomograms for quadratic equations for any given ranges of the coefficients.

EXAMPLES IV.

1. Construct nomograms for the equations:

(i) $x^2 - ax + b = 0$, (ii) $x^4 + ax^2 + b = 0$;

(iii) $ax^2 + bx + 1 = 0$; (iv) $x^2 = ax + b$;

(v) $ax + \frac{b}{x} = 1$; (vi) $x^{\frac{1}{2}} + \frac{a}{x^{\frac{1}{2}}} = b$.

Note.—In each case find a transformation which puts the equation in the form $x^2 + ax + b = 0$, and use the method of § 41.

2. Construct a nomogram for $x^2 + ax + b = 0$, as in § 41, and use it to solve the following equations.

(i) $x^2 + 2x + \frac{1}{2} = 0$; (ii) $3x^2 - x - \frac{1}{4} = 0$;

(iii) $-\frac{1}{2}x^2 + x + 1 = 0$; (iv) $\frac{1}{x} + x = 7$;

(v) $x^2 + 17x - 42 = 0$; (vi) $\frac{x^2}{100} + \frac{3x}{10} + 1 = 0$.

3. Construct nomograms for :

(i) $x^2 + \frac{a}{x} = b$; (ii) $x^4 = ax + b$; (iii) $x^3 = a + bx$;

(iv) $x^{\frac{1}{2}} = \frac{a}{x} + b$; (v) $x^{\frac{1}{3}} = a + bx$; (vi) $x^{\frac{4}{3}} = \frac{a}{x} + b$.

4. With a nomogram for $x^3 + ax + b = 0$, solve the equations :

(i) $x^3 - 3x + 2 = 0$; (ii) $x^3 + 6x^2 - 5x - 7 = 0$;
(iii) $x^3 - 4x^2 + 1 = 0$; (iv) $x^3 - 5x^2 - 11x + 7\frac{1}{2} = 0$.

5. Construct nomograms for the equations in question 1, using the method of § 47.

6. Use a nomogram for $x^2 + ax + b = 0$, according to § 47, to solve the equations in question 2.

7. Construct a nomogram for $x^2 + ax + b = 0$, in which a can have values between 0 and 100 and b between 0 and -100.

8. Shew that the nomogram in question 7 can be used for both ranges 0 to -100 by putting $x = -x'$.

9. Construct nomograms for $x^2 + ax + b = 0$, for the following ranges of a, b respectively :

(i) 0 to 10, 0 to -1000 , (ii) 0 to 1000, 0 to -10 ;
(iii) 0 to 50, -1000 to $-10,000$;
(iv) 0 to 50, 1000 to 10,000

10. In direct fire the distance y that the shell has risen in time t is given by $y = \frac{1}{2}gt(T - t)$, where T is the whole time of flight for a range on a horizontal plane. Shew how to use the quadratic equation nomogram to find y when t and T are given Take e g.

(i) $T = 10$, $t = 5$; (ii) $T = 17\frac{1}{4}$, $t = 3{\cdot}2$; (iii) $T = 27$, $t = 25$

11. In experiments on coal-gas combustion, the flame temperature is given by the equation $t^2 - at + b = 0$ where a, b range between 4,000 to 12,000, 8,000,000 to 25,000,000 respectively. Construct a suitable nomogram

12. Construct a nomogram for $x^3 + ax^2 + b = 0$, using graduation on two parallel scales.

CHAPTER V

GENERAL THEORY OF NOMOGRAMS WITH TWO PARALLEL SCALES. PARALLEL COORDINATES

51. In Chapter IV. we have seen that nomograms can be made for quadratic and other equations in which the coefficients are represented by points on uniformly graduated parallel scales. We have also seen that the mathematical part of the work can be greatly diminished if we *assume* that a nomogram exists. We shall return in a later chapter (Chapter VIII.) to this automatic method; in the present chapter we shall consider the theory of such nomograms in general.

Distances a, b cut off on parallel uniform scales are called **parallel coordinates.**

52. Nomograms with Parallel Coordinates.

We shall then proceed to shew how to find the equation of the x scale in a nomogram with parallel coordinates, and how to graduate it

Choose rectangular axes $O\xi$, $O\eta$ (Fig. 46), so that $O\eta$ is the b scale and O is the zero of the b scale. Let the a scale be at a distance of one ξ unit to the left (negative side) of the b scale. Consider any point P

whose coordinates are (ξ, η). We shall find what relation there must be between the lengths a, b cut off on the a, b scales for all lines that pass through P.

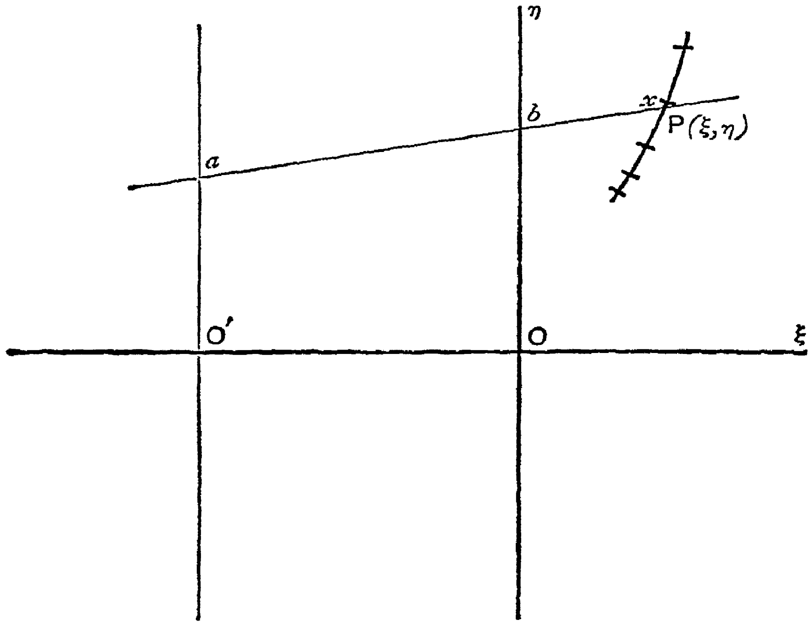


FIG 46

Take the line through P given by a, b. The equation of this line is

$$Y = b + (b - a)X,$$

where (X, Y) are the coordinates of any point on it. Since P is on this line this equation also holds for (ξ, η) the coordinates of P itself. This is true for *all* lines through P. Hence the a, b intercepts for all lines through P satisfy the single equation

$$\eta = b + (b - a)\xi,$$

i.e.

$$\frac{\xi}{\eta} a - \frac{\xi + 1}{\eta} b + 1 = 0.$$

Let then the equation, given to be solved, be written

$$A(x)a+B(x)b+1=0,$$

so that $A(x)$ and $B(x)$ are known functions of x. If there is to be a definite graduation x at P it follows that the two equations

$$\frac{\xi}{\eta}a-\frac{\xi+1}{\eta}b+1=0$$

and

$$A(x)a+B(x)b+1=0,$$

are *both* true for *any number* of straight lines through P. This means that the equations are really the same, so that we must have

$$A(x)=\frac{\xi}{\eta},\quad B(x)=-\frac{\xi+1}{\eta},$$

i.e.

$$\xi=-\frac{A(x)}{A(x)+B(x)},\quad \eta=-\frac{1}{A(x)+B(x)}.$$

If we eliminate x we obtain an equation between ξ, η, telling us what the relation between the coordinates of P must be if P is to be a graduation in the nomogram, *i.e.* a point on the x curve. This relation defines the x curve, which can be plotted. The graduation of the curve is effected by finding from the above equations what is the value x for any point (ξ, η). In practice, it is often simpler to consider x as a parameter in terms of which ξ, η are given, this parameter being in fact the graduation.

53 Ex.

Let us go through this reasoning with a specific instance. Take d'Ocagne's nomogram for the quadratic equation

$$x^2+ax+b=0.$$

If we write this equation in the form

$$\frac{1}{x}a+\frac{1}{x^2}b+1=0,$$

we get for the coordinates (ξ, η) of any point P on the x curve of the nomogram :

$$\frac{1}{x}=\frac{\xi}{\eta},\quad \frac{1}{x^2}=-\frac{\xi+1}{\eta}.$$

Hence the equation of the x curve is

$$\left(\frac{\xi}{\eta}\right)^2=-\frac{\xi+1}{\eta},$$

i.e.

$$\eta=-\frac{\xi^2}{\xi+1}.$$

This is actually the equation used above (§ 47, Fig. 40). The x graduation at any point is given by

$$x=\frac{\eta}{\xi}=-\frac{\xi}{\xi+1},$$

as used in § 47.

54. The Parabolic Nomogram.

In § 52 we have made the b scale coincide with the y axis and the a scale along the line $\xi=-1$. We are of course at liberty to choose any two parallel lines for these scales. Let us then make the a scale along the line $\xi=-1$ and the b scale along the line $\xi=+1$.

For *all* lines through any given x graduation the a, b intercepts must satisfy the given equation

$$A(x)a+B(x)b+1=0.$$

But the equation of any line through this point, coordinates (ξ, η), must also be

$$\eta=\frac{b+a}{2}+\frac{b-a}{2}\xi,$$

i.e. $$\frac{\xi-1}{2\eta}a-\frac{\xi+1}{2\eta}b+1=0.$$

Hence the coordinates (ξ, η) of the point graduated x on the x curve are given by

$$A(x)=\frac{\xi-1}{2\eta}, \quad B(x)=-\frac{\xi+1}{2\eta}.$$

If now we have the quadratic equation

$$x^2+a'x+b'=0,$$

and use the values a, b defined by

$$a=b'-a', \quad b=b'+a',$$

we obtain $$x^2+\frac{b-a}{2}x+\frac{b+a}{2}=0,$$

so that the equation can be written

$$\frac{1-x}{2x^2}a+\frac{1+x}{2x^2}b+1=0.$$

Thus we have $$A(x)=\frac{1-x}{2x^2}, \quad B(x)=\frac{1+x}{2x^2},$$

and we get $$\frac{1-x}{2x^2}=\frac{\xi-1}{2\eta}, \quad \frac{1+x}{2x^2}=-\frac{\xi+1}{2\eta}.$$

This at once yields $x=\xi, \quad \eta=-x^2=-\xi^2.$

Thus the x curve is the parabola $\eta=-\xi^2$, and the x graduation at any point on this parabola is the corresponding abscissa ξ. This is the nomogram given above, Ch. IV., § 41, Fig. 38.

55 The Circular Nomogram

A particularly interesting nomogram for the quadratic equation can be obtained, in which the x curve is a circle.

Let the a, b scales be along the axis of η, and the line $\xi=+1$, respectively. Any line through the point (ξ, η) cuts off from the a, b scales lengths which satisfy the equation

$$\eta=a+(b-a)\xi,$$

i.e.
$$\frac{\xi-1}{\eta}a-\frac{\xi}{\eta}b+1=0.$$

If the quadratic equation

$$x^2+a'x+b'=0$$

is written in the form

$$x^2-\frac{x}{a}+\frac{b}{a}=0,$$

so that
$$a=-\frac{1}{a'},\quad b=-\frac{b'}{a'}$$

are the definitions of a, b in terms of a', b', the equation for x can be written

$$-ax-\frac{b}{x}+1=0.$$

This is the same equation as

$$\frac{\xi-1}{\eta}a-\frac{\xi}{\eta}b+1=0,$$

if
$$x=-\frac{\xi-1}{\eta},\quad x=\frac{\eta}{\xi},$$

so that we get for the x curve the equation

$$\frac{\eta}{\xi}=-\frac{\xi-1}{\eta},$$

i e.
$$\xi^2+\eta^2-\xi=0,$$

or
$$(\xi-\tfrac{1}{2})^2+\eta^2=(\tfrac{1}{2})^2.$$

The x curve is therefore the circle whose centre is the point $(\frac{1}{2}, 0)$ and whose radius is $\frac{1}{2}$. The graduations are at once given by $x=\eta/\xi$.

In Fig. 47 the circle is shewn drawn. To solve the equation

$$x^2+a'x+b'=0,$$

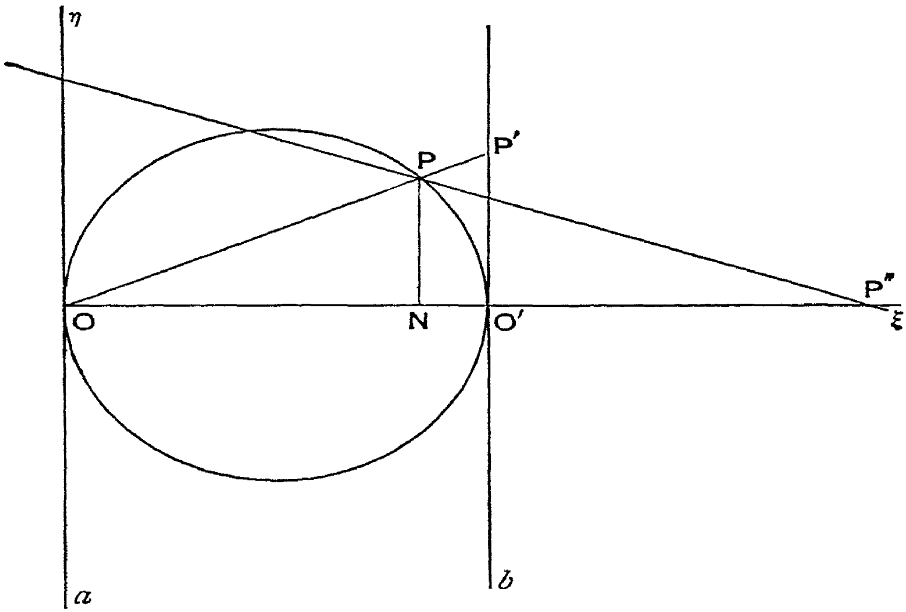


FIG 47.

take the distance $-\frac{1}{a'}$ along the a scale from its point of contact with the circle, and the distance $-\frac{b'}{a'}$ along the b scale from its point of contact with the circle. The join cuts the circle at, say, P, whose x graduation is η/ξ, *i.e.* the tangent of the angle $PO\xi$.

This interesting case can be interpreted geometrically in a very simple manner. Let P (Fig. 47) be a point on the circle. Join OP and produce it till it meets the b scale in P'; drop PN perpendicular to the diameter OO'. Then we have $\xi = ON$, $\eta = NP$, so that $x = \tan POO'$.

But OO' is unit distance: hence x is numerically equal to the distance $O'P'$. Thus the circle is readily graduated by joining O to the graduations of the uniform b scale, and letting the joins cut the circle.

Also let the line through the points a, b cut the ξ axis at the point P''. It is clear that

$$O'P''/P''O = -b/a = -b'.$$

Thus the η axis is graduated *reciprocally and negatively* $\left(a = -\frac{1}{a'}\right)$, and the ξ axis is graduated *segmentally and negatively*. The former needs no explanation. The latter should be compared to the segmentary scale (b), described in Ch. I., § 10, Fig. 10. There only internal divisions are taken, whereas here we graduate both internally and externally, the internal graduations being called negative, and the external ones positive, *i.e.* a minus sign is added to the ordinary geometrical convention. It is important to remember that the η unit must be equal to the ξ unit, *i.e.* to the diameter of the circle.

The line joining a' on η to b' on ξ cuts the circle at points whose graduations are the roots of the quadratic equation

$$x^2 + a'x + b' = 0.$$

In Fig. 48 only half the circle is drawn, since we only need to find a positive root.

This nomogram (due to Whittaker) is remarkable because of the ease with which the x curve can be drawn and graduated. On the other hand the com-

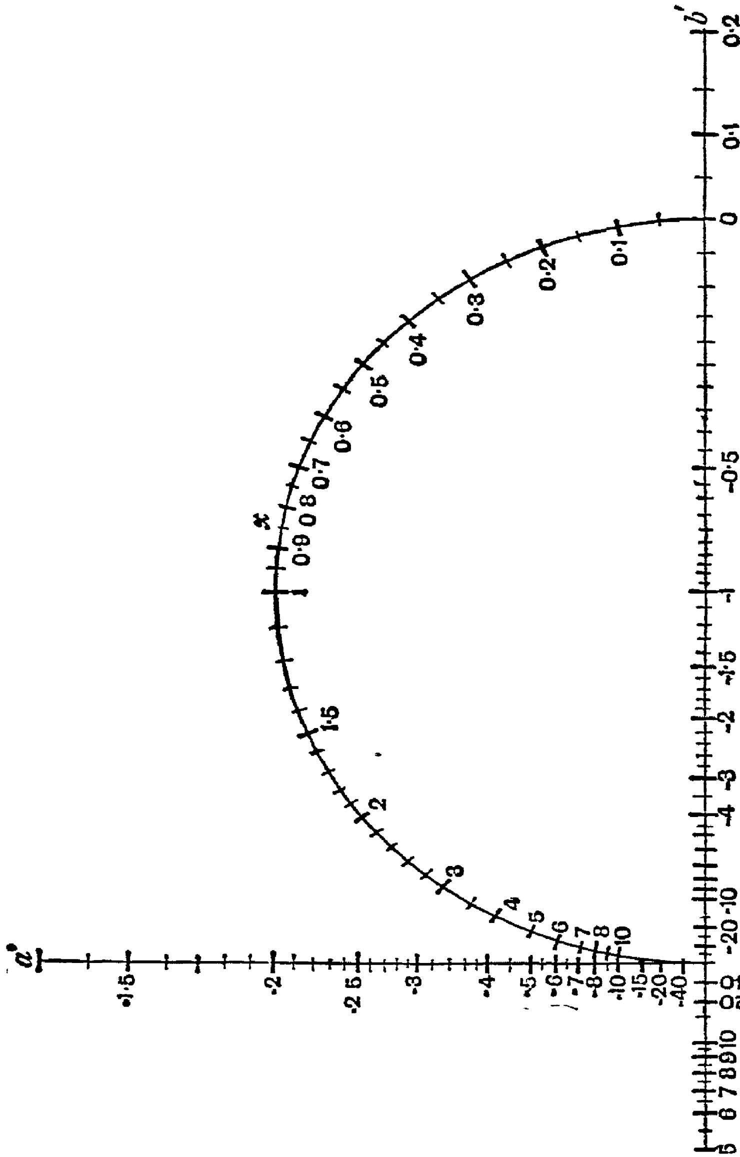


FIG. 44.

plication in the graduations on the a', b' scales is a distinct disadvantage. We shall return to it later, Ch. VII., § 68, as it is an instance of a nomogram in which parallel coordinates are no longer used.

56. Cubic Equation.

Let the cubic equation (of Cardan's form)

$$x^3+ax+b=0$$

be written in the form

$$\frac{a}{x^2}+\frac{b}{x^3}+1=0.$$

Using for a, b scales the lines $\xi=-1$, $\xi=0$, we have, if (ξ, η) are the coordinates of the graduation x, the relation (§ 52)

$$\frac{\xi}{\eta}a-\frac{\xi+1}{\eta}b+1=0.$$

Hence

$$\frac{\xi}{\eta}=\frac{1}{x^2}, \quad \frac{\xi+1}{\eta}=-\frac{1}{x^3},$$

so that

$$\left(\frac{\xi}{\eta}\right)^3=\left(\frac{\xi+1}{\eta}\right)^2,$$

giving

$$\eta=\frac{\xi^3}{(\xi+1)^2},$$

as the equation of the x curve; the graduations are given by

$$x=\sqrt{\frac{\eta}{\xi}}.$$

For plotting purposes we use

$$\xi=-\frac{x}{1+x}, \quad \eta=-\frac{x^3}{1+x}.$$

The student is advised to plot this curve and to graduate it; he will find it convenient to take the ξ unit ten times the η unit.

Here, too, we modify the fundamental nomogram in accordance with the ranges of a and b (see Ch. IV., §§ 48-50).

EXAMPLES V.

1. Using parallel coordinates a, b along the line $\xi = -1$ and the line $\xi = 0$, construct nomograms for the following equations :

(i) $ax^2 + bx + 1 = 0$; (ii) $ax + \frac{b}{x} + 1 = 0$;

(iii) $x^3 - ax + b = 0$; (iv) $x^3 + ax^2 + b = 0$;

(v) $a\left(x + \frac{1}{x}\right) + b\left(x - \frac{1}{x}\right) = 1$; (vi) $x^4 + ax + b = 0$.

2. Do the question 1 with parallel coordinates along the lines $\xi = -\frac{1}{2}$, $\xi = +\frac{1}{2}$.

3. Try the equation $ax + \frac{b}{x} + 1 = 0$ with the parallel coordinates measured along the following pairs of lines :

(i) $\xi = 0$, $\xi = 1$; (ii) $\xi = \frac{1}{2}$, $\xi = -\frac{1}{2}$; (iii) $\xi = -1$, $\xi = 0$.

Why is the second the easiest process ?

4. Try the equation $x^2 + ax + b = 0$ with parallel coordinates along the lines $\xi = \alpha$, $\xi = \beta$. Deduce that

$$\xi = \frac{\alpha x + \beta}{x + 1}, \quad \eta = -\frac{x^2}{x + 1},$$

and decide which are the most convenient values of α, β to choose. (*They must be unequal, of course.*)

5. Carry out the investigation of question 4 for the equation $ax^2 + bx + 1 = 0$.

6. Do the same for the equation $ax + \frac{b}{x} = 1$.

7. If you have to construct a nomogram for $x^2 = \frac{1}{ax + b}$, where would you place the a, b scales ?

8. The volume V cubic feet of water in a hemispherical pot of radius a feet, filling it to a depth x feet, is given by $V = \pi\left(ax^2 - \frac{x^3}{3}\right)$. Construct a nomogram to find x for any values of a, V.

9. The resistance R of a sphere moving through a gas is given by

$$R = AU^2 + BU,$$

where U is the velocity and A, B depend on the size of the sphere and on the nature of the gas Construct a nomogram to find what velocity U will give a definite prescribed resistance R.

CHAPTER VI

NOMOGRAMS WITH TRIGONOMETRICAL FUNCTIONS

57. There being no restriction on the forms of $A(x)$, $B(x)$ in the method of the last chapter, we can use this method for the construction of nomograms for transcendental equations; as, *e.g.*, equations involving trigonometrical functions.

58. Nomogram for $a \tan x + b \sec x + 1 = 0$.

In order to decide on the best positions of the a, b scales, let us put the a scale on the line $\xi = \alpha$, the b scale on the line $\xi = \beta$, in the method of the last chapter. The equation of the line joining the a graduation to the b graduation is

$$\eta = \frac{b-a}{\beta-\alpha}(\xi-\alpha)+a,$$

i.e.

$$\frac{\xi-\beta}{\beta-\alpha}\frac{a}{\eta}-\frac{\xi-\alpha}{\beta-\alpha}\frac{b}{\eta}+1=0.$$

To make this agree with

$$a \tan x + b \sec x + 1 = 0,$$

we make

$$\frac{\xi-\beta}{(\beta-\alpha)\eta}=\tan x, \quad \frac{\xi-\alpha}{(\beta-\alpha)\eta}=-\sec x.$$

We at once get, since $\sec^2 x = 1 + \tan^2 x$, the equation

$$(\xi - a)^2 - (\xi - \beta)^2 = (\beta - a)^2 \eta^2,$$

i.e.

$$2\xi = (\beta - a)\eta^2 + (\beta + a).$$

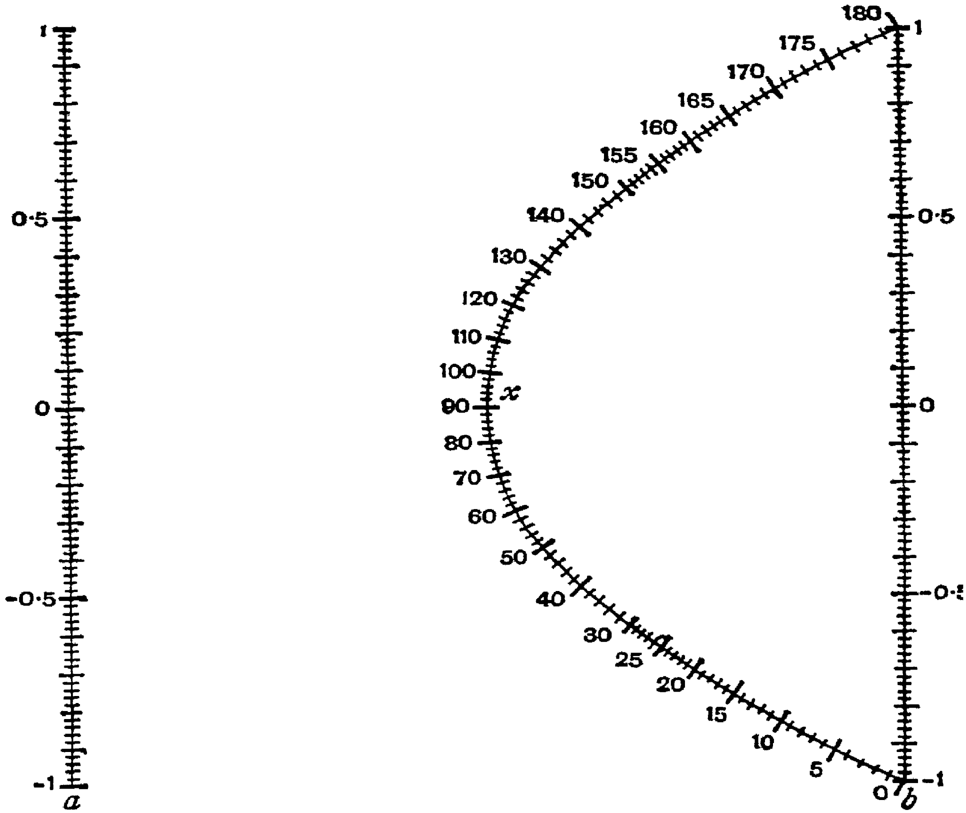


FIG. 49.

To get the simplest result, we therefore use

$$a + \beta = 0.$$

This gives $$2\xi = 2\beta\eta^2.$$

Hence we choose $\beta = 1, \quad a = -1,$

and the x curve is the parabola $\xi = \eta^2$.

The graduations are seen to be given by

$$\sin x = \frac{1 - \xi}{1 + \xi}.$$

(See Fig. 49, which is applicable to all really useful values of a, b.)

59. Nomogram for $a' \sin x + b' \cos x = c'$.

Write this in the form

$$\frac{a'}{b'}\tan x - \frac{c'}{b'}\sec x + 1 = 0,$$

and we have the same problem as in § 58, the numbers $\frac{a'}{b'}$, $-\frac{c'}{b'}$ being used on the a, b scales respectively in Fig. 49.

It is, of course, easy to make a nomogram for

$$a \sin x + b \cos x = 1,$$

in which we would use $\frac{a'}{c'}$ for a, $\frac{b'}{c'}$ for b, in order to solve the equation

$$a' \sin x + b' \cos x = c'.$$

If we use the method of § 58, we get

$$\frac{\xi - \beta}{(\beta - \alpha)\eta} = -\sin x, \quad \frac{\xi - \alpha}{(\beta - \alpha)\eta} = \cos x.$$

Hence, since $\sin^2 x + \cos^2 x = 1$, we get

$$(\xi - \alpha)^2 + (\xi - \beta)^2 = (\beta - \alpha)^2\eta^2,$$

i.e. $$2\xi^2 - 2\xi(\alpha + \beta) + \alpha^2 + \beta^2 = (\beta - \alpha)^2\eta^2.$$

The x curve is simplified if we choose once more

$$\alpha + \beta = 0, \quad \textit{i.e.}\ \alpha = -\beta,$$

and the x curve becomes

$$\frac{\eta^2}{\frac{1}{2}} - \frac{\xi^2}{\beta^2} = 1,$$

a hyperbola. We might use, *e.g.*, $\beta^2 = \frac{1}{2}$; the graduations are then given by

$$\tan x = \frac{1/\sqrt{2} - \xi}{1/\sqrt{2} + \xi}.$$

60. If the student will plot the x curve in § 59, he will find that the resulting nomogram is not convenient,

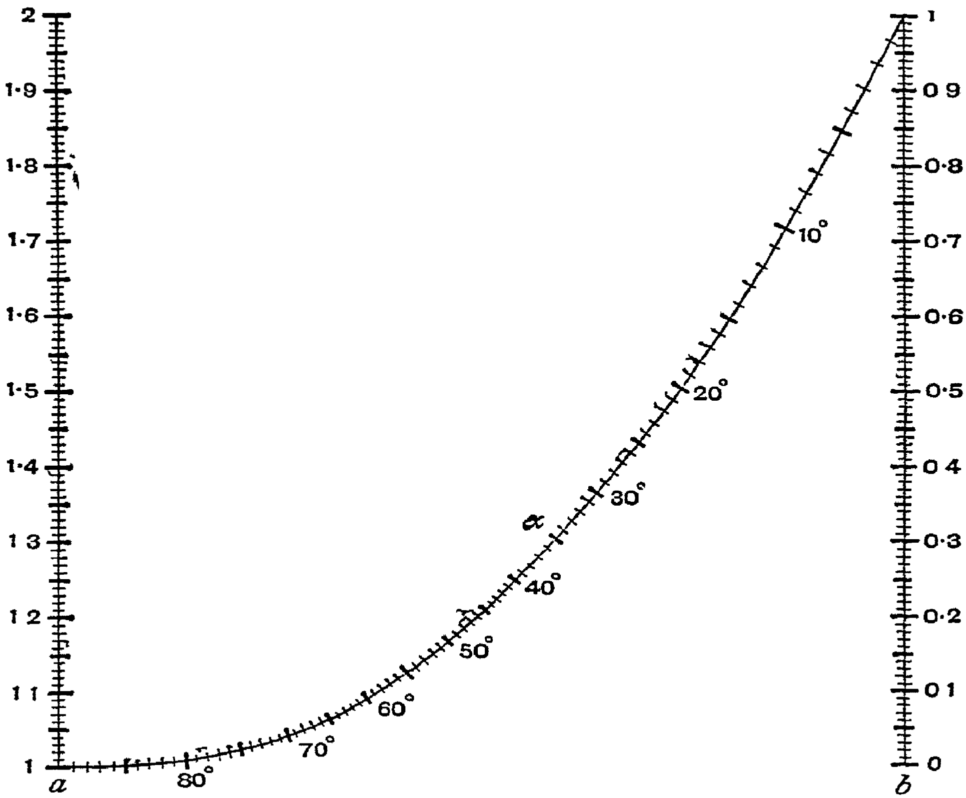


FIG 50.

since the x curve is a more or less shallow curve lying symmetrically between the scales, and this means rather inaccurate readings. To remedy this we again use the idea of oblique axes developed in the case of the quadratic equation in Ch. IV. This is really equivalent to shifting the b scale in one direction along itself, say upwards, and the a scale in the opposite direction.

In Fig. 50 we have carried out this idea, and we have a nomogram very well adapted for values of a between 1 and 2, and values of b between 0 and 1.

61. Kepler's Equation: $nt = \phi - e \sin \phi$.

In a very important problem in astronomy, it is necessary to solve the equation

$$nt = \phi - e \sin \phi,$$

in which nt and ϕ are certain angles (in radians), and e is the eccentricity of the orbit of a planet. To find ϕ nomographically when nt and e are given, we have to construct a nomogram in which the same quantity ϕ occurs algebraically and trigonometrically. This does not affect the applicability of the method of Chapter V. We use the method of § 58 to find the best positions of the scales.

In practice e will always be between 0 and 1 (actually the useful limits of e are much closer together: 0 and 0·4); whilst nt will go through the values 0 to π (the range π to 2π being obviously obtainable from the values 0 to π). Hence the nt range is several times longer than the e range. We would therefore find it advisable to take the e unit, say 5 times the nt unit.

Let then $a = nt$, $b = 5e$, so that the equation is

$$a + \frac{b}{5} \sin \phi = \phi.$$

We get by the method of § 58 :

$$\frac{\xi-\beta}{(\beta-\alpha)\eta}=-\frac{1}{\phi}, \quad \frac{\xi-\alpha}{(\beta-\alpha)\eta}=\frac{\sin\phi}{5\phi}.$$

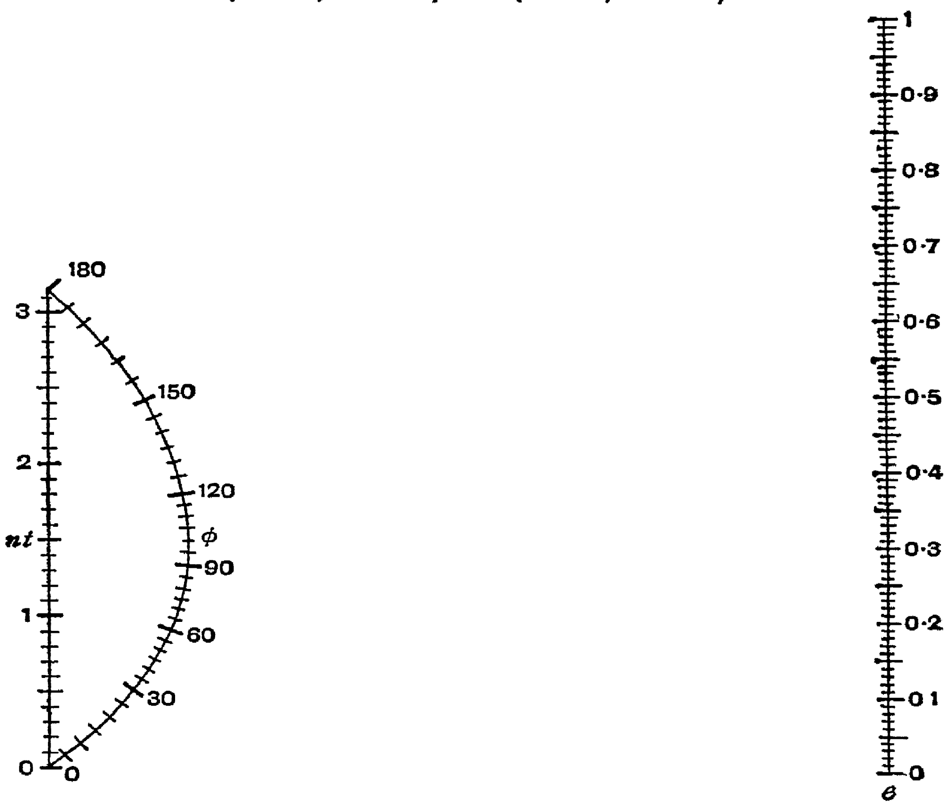


Fig 51.

There is no very obvious indication of most convenient values of α, β. We therefore use ones giving least arithmetic ; *e.g.* $\beta=0, \quad \alpha=-1,$

as in d'Ocagne's nomogram for the quadratic equation.

We get $$\frac{\xi}{\eta}=-\frac{1}{\phi}, \quad \frac{\xi+1}{\eta}=\frac{\sin\phi}{5\phi}.$$

Hence $$\frac{\xi+1}{\xi}=-\frac{\sin\phi}{5},$$

so that $$\xi = -\frac{5}{\sin\phi + 5},$$

and therefore $$\eta = \frac{5\phi}{\sin\phi + 5}.$$

The graduations are given by the simple relation

$$\phi = -\eta/\xi,$$

which has an obvious geometrical interpretation.

The nomogram is given in Fig. 51, and the student should verify it by tracing it on squared paper.

62. Nomogram for $\frac{1}{2}$ab sin C.

The nomograms in Ch. III. can be modified so as to include problems of products involving trigono-

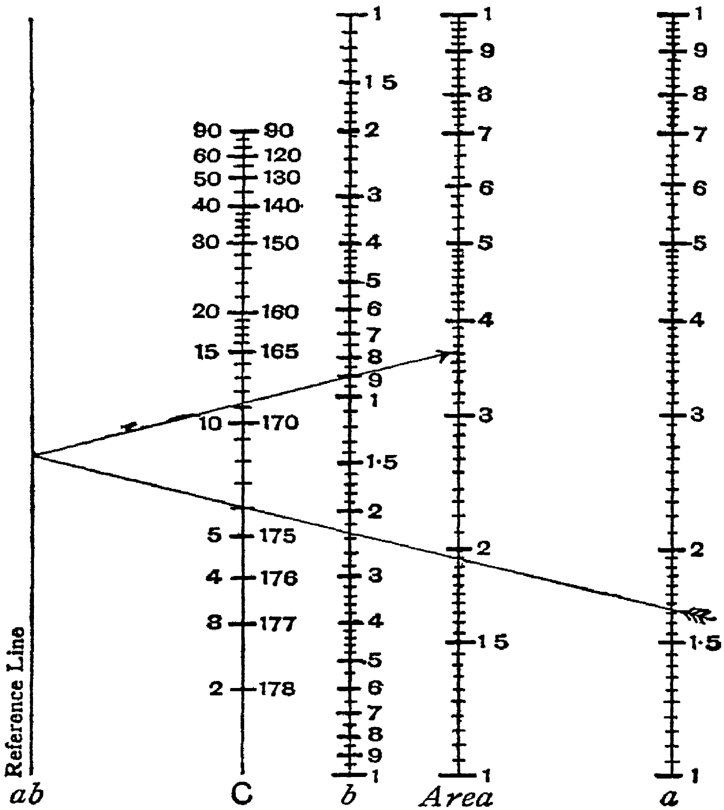


FIG. 52.

metrical functions. Let us take, as an illustration, the formula for the area of a triangle in terms of two sides and the included angle. If we take the nomogram for the product of three quantities (Fig. 31) and use a instead of A, b instead of B, and $\frac{1}{2}\sin C$ instead of C, we get the required nomogram for $\frac{1}{2}ab\sin C$ (Fig. 52).

63. Nomogram for $\tan\frac{B-C}{2}=\frac{b-c}{b+c}\cot\frac{A}{2}$.

Again, consider the well-known formula for finding the angles of a triangle in which two sides b, c and the

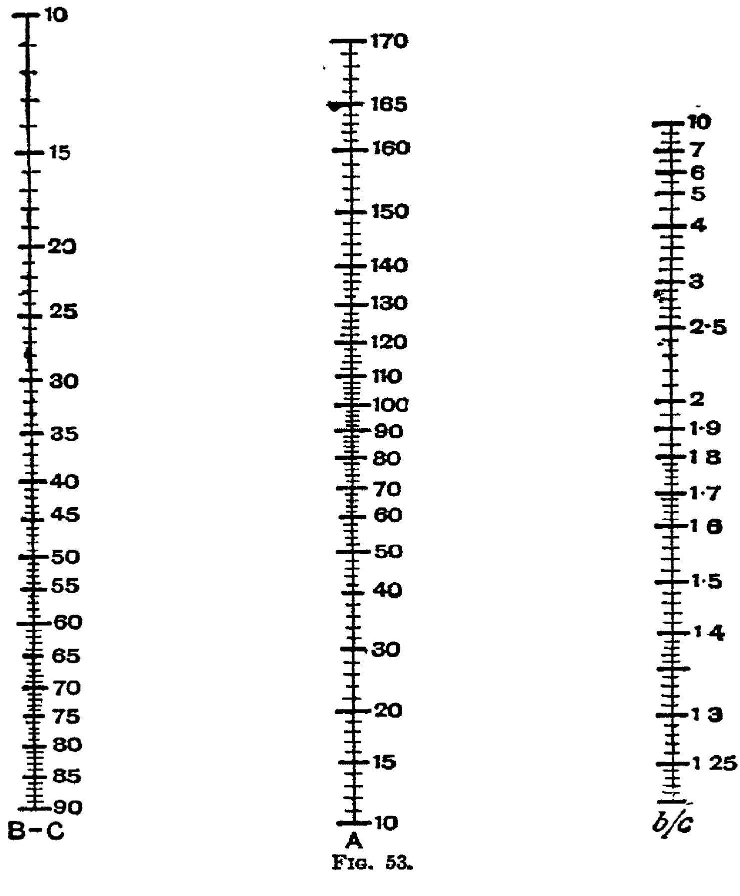


FIG. 53.

included angle A are given. Writing the formula as

$$\tan\frac{B-C}{2}=\frac{\frac{b}{c}-1}{\frac{b}{c}+1}\cot\frac{A}{2},$$

we see that the quantities b and c only enter in terms of the ratio b/c. We can assume $b > c$. If we write

$$X=\tan\frac{B-C}{2},\quad Y=\frac{b-c}{b+c},\quad Z=\cot\frac{A}{2},$$

the nomogram for $X=YZ$ will give the nomogram required when regraduated. (Fig. 53, which the student should compare with Fig. 28.)

Examples VI.

1. Construct nomograms for

(i) $\sin x=a+b\cos x$; (ii) $\sec x=a+b\tan x$;
(iii) $a\sin 2x+b\sin x+1=0$; (iv) $\cos 2x=a\sin x+b$;
(v) $\sin x=a+bx$; (vi) $\tan x=a+bx$.

In (v) and (vi) let a range between ± 1 and b between ± 4.

2. Construct nomograms for

(i) $\frac{a}{b}\sin B$; (ii) $\frac{a}{2\sin A}$; (iii) $s\tan\frac{A}{2}$;

(iv) $r\frac{\sin\alpha}{\alpha}$; (v) $2\pi a^2(1-\cos\alpha)$.

3. Construct nomograms for

(i) $\frac{v}{32}=\frac{m_1-m_2}{m_1+m_2}t$; (ii) $f=\frac{32m_1\sin\alpha}{m_1+m_2}$;
(iii) $f=32(\sin\alpha-\frac{1}{3}\cos\alpha)$.

4. Shew that the nomograms in §§ 58-60 can be derived from nomograms for quadratic equations.

5. If $1/D$ represents the thickness of a thread in inches, and n the number of turns per inch, the angle of twist, θ, is given by $D/\pi\cot\theta=n$. Construct a nomogram in which D ranges from 5 to 200 and θ from 0° to 45°.

CHAPTER VII

NOMOGRAMS WITH INTERSECTING SCALES

64. If we refer back to the circular nomogram for the quadratic equation, we shall see that in the end it was found best to graduate the axes of ξ, η, *i.e.* the line on which a solution of the quadratic lies is taken through two points on lines perpendicular to one another. It is often convenient to use such scales, or even scales at some other angle with one another. We shall first investigate the theory and practice of nomograms with perpendicular scales, and then proceed to the generalised method.

65. Nomogram for the Optical Formula.

A useful case is that of the optical formula

$$\frac{1}{f}=\frac{1}{u}+\frac{1}{v},$$

where f is the focal length of a lens and u, v are the distances of object and image measured on opposite sides. To find one of the quantities u, v, f, given the other two, we can proceed as follows.

Take two axes $O\xi$, $O\eta$ at right angles to one another, and draw the line bisecting the angle between them. Graduate $O\xi$, $O\eta$ uniformly with the same unit and call them u, v respectively. Graduate the bisector with unit $\sqrt{2}$ times as great and call it f. Then

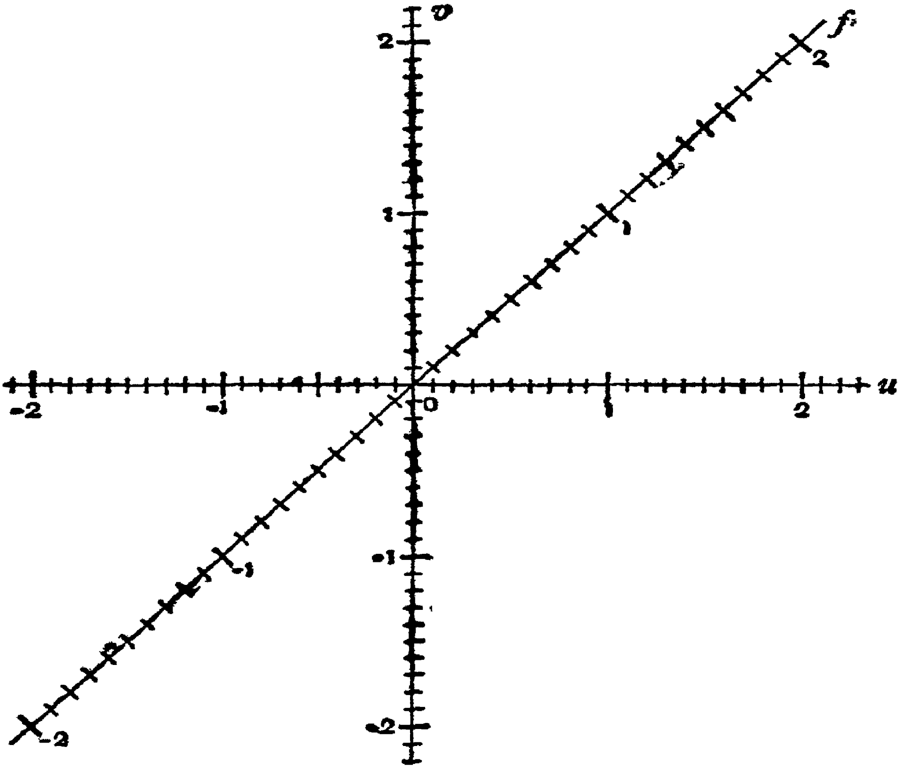


FIG. 54.

three collinear graduations u, v, f satisfy the above relation (Fig. 54).

66. Nomogram for the Relation between the Elastic Coefficients of a Body.

If E is the modulus of elasticity of a body, K its bulk modulus and C the modulus of rigidity, then we have

$$\frac{1}{E}=\frac{1}{3C}+\frac{1}{9K}.$$

To construct a nomogram for this relation we take the diagram of Fig. 54, and substitute E for f, multiply the unit of u by 3 and call the u scale C, multiply the unit of v by 9 and call the v scale K. Then three collinear graduations E, C, K satisfy the above relation.

67. Nomograms with Perpendicular Scales.

Let the scales be taken along the axes of ξ, η respectively. Let P be a point (ξ, η) on the x curve

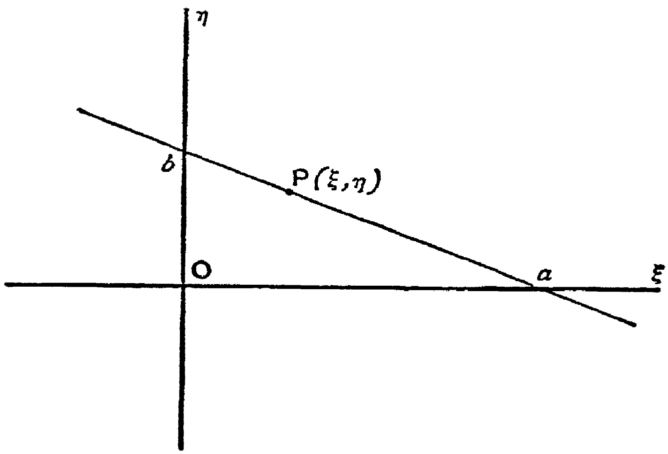


FIG 55.

of a nomogram. Take a line through P (Fig. 55), and let the intercepts on the axes be a, b. Then, since the line passes through the point (ξ, η), we have the relation

$$\frac{\xi}{a}+\frac{\eta}{b}=1,$$

and this is satisfied by *all* lines through P. If, then, we have an equation which can be written in the form

$$\frac{A(x)}{a}+\frac{B(x)}{b}+1=0,$$

we get a solution at P on the line a, b if the two equations are identical—because these relations must hold for an indefinite number of pairs of intercepts a, b. We therefore have

$$A(x) = -\xi, \quad B(x) = -\eta,$$

which gives us the x curve and also the graduations. It is often convenient to plot the x curve straightforwardly from the equations

$$\xi = -A(x), \quad \eta = -B(x),$$

looking on x as a parameter : it is also the graduation on the x curve.

68. The Circular Nomogram.

Let us see, *e.g.*, how Whittaker's circular nomogram for the quadratic equation is obtained with the perpendicular scales used in Fig. 48. Let the equation be

$$x^2 + a'x + b' = 0,$$

and write it in the form

$$x^2 - \frac{x}{b} + 1 - \frac{1}{a} = 0.$$

This can be written

$$-\frac{1}{1+x^2}\frac{1}{a} - \frac{x}{1+x^2}\frac{1}{b} + 1 = 0.$$

Comparing with $$\frac{\xi}{a} + \frac{\eta}{b} = 1,$$

we see that $$\xi = \frac{1}{1+x^2}, \quad \eta = \frac{x}{1+x^2}.$$

This gives $\xi^2 + \eta^2 = \xi$ and $x = \eta/\xi$,

the equation and graduation used in Ch. V. § 55, Figs. 47, 48.

The graduations of the a, b scales are given by

$$a=\frac{1}{1-b'}, \quad b=-\frac{1}{a'},$$

which agree with the graduations used in § 55.

69. Nomogram for $x=a^b$.

Another interesting nomogram with perpendicular scales is that for *any number raised to any power.*

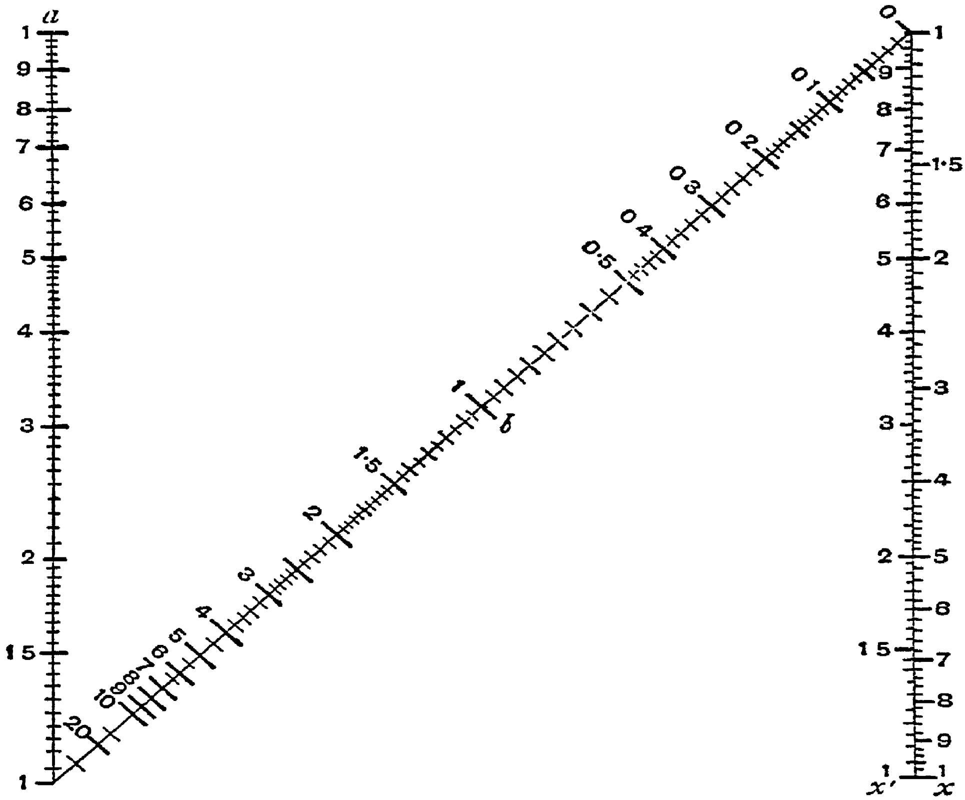


FIG 56

Construct a logarithmic scale a along the axis of η, and a segmentary scale b on the ξ axis between the origin and the point O' ($\xi=1$), so that each graduation is the ratio of its distances from O', O respectively. At O' on the line $\xi=1$ construct a logarithmic scale, x, exactly like that on the η axis, but in the opposite direction. Then we get for collinear graduations

$$\log x = b \log a,$$

giving $x=a^b$. If we also graduate x' on the x scale, but backwards, we get

$$\log x' = -b \log a,$$

giving $x'=a^{-b}$. If both sets of graduations are given, *any power, positive or negative, of any quantity* can be found. The remarkable simplicity of this nomogram makes it very convenient when great accuracy is not required. In Fig. 56 we have made a more convenient nomogram, using oblique axes, as in § 48.

70. Nomograms with Intersecting Scales at Any Angle.

We can justify the suggestion at the end of § 69, as follows. Let the scales be drawn on lines $O\xi$, $O\eta$ defining a system of oblique coordinates. If $P(\xi, \eta)$ in Fig. 57 is a point x on the x curve of a nomogram, let a line through P cut off intercepts a, b from the axes at the points P', P'', so that $OP'=a$, $OP''=b$, measured positive and negative, with the usual convention. Draw PM, PN parallel to the axes. Then $OM=\xi$, $ON=\eta$.

Now $$\frac{\xi}{a}=\frac{OM}{OP'}=\frac{NP}{OP'}=\frac{P''P}{P''P'},$$

and $$\frac{\eta}{b}=\frac{ON}{OP''}=\frac{PM}{P''O}=\frac{PP'}{P''P'};$$

hence $$\frac{\xi}{a}+\frac{\eta}{b}=\frac{PP'+P''P}{P''P'}=\frac{P''P'}{P''P'}=1.$$

Thus *all* lines through P have a, b satisfying the equation

$$\frac{\xi}{a}+\frac{\eta}{b}=1.$$

If then x is a solution of the equation

$$\frac{A(x)}{a}+\frac{B(x)}{b}+1=0,$$

we have $$A(x)=-\xi,\quad B(x)=-\eta,$$

as in the case of rectangular coordinates.

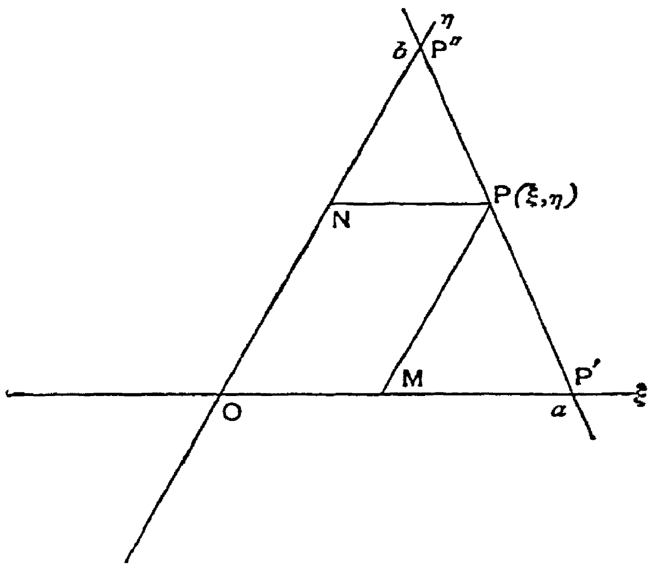


FIG 57.

We see, therefore, that the general method is just as simple as the method with perpendicular scales. With oblique coordinates the plotting will as a rule be more difficult owing to the non-availability of rhombussed paper: in practice we may use the method sketched in §§ 49-50, to be dealt with in greater detail in the next Chapter.

71. Nomogram for Optical Formula with Equal Scales.

Putting $\xi=\eta=-A(x)=-B(x)=x$

in § 70, we get

$$\frac{1}{a}+\frac{1}{b}=\frac{1}{x},$$

the optical formula. The student will easily convince himself that if we construct the nomogram of Fig. 54 with an angle of 120° between the ξ, η axes, then the three scales will all have the same unit. This is from some points of view a rather useful property.

EXAMPLES VII.

1. Construct nomograms for

(i) $\frac{1}{f}=\frac{1}{u-1}+\frac{1}{v-2}$; (ii) $\frac{1}{f}=0{\cdot}622\left(\frac{1}{r}-\frac{1}{s}\right)$;

(iii) $\frac{1}{F}=\frac{1}{f_1}+\frac{1}{f_2}+\frac{1}{f_3}$; (iv) $\frac{1}{t}=\frac{1}{2}\left(\frac{1}{r}-\frac{1}{s}\right)+\frac{1}{t'}$.

These formulae are adaptations of familiar ones in optical work. To add up three or more reciprocals, as in (iii) and (iv), we use reference lines as in Ch. III.

2. Construct a nomogram for $\frac{1}{f}=\frac{1}{u}-\frac{1}{v}$, using parallel coordinates. Compare the advantages and disadvantages of this nomogram and the nomogram for the optical formula given in the text.

3. Shew that, by the use of the idea in § 71, it is possible to construct a nomogram for

$$\frac{1}{F}=\frac{1}{f_1}+\frac{1}{f_2}+\frac{1}{f_3}+\frac{1}{f_4}+\dots$$

to any number of reciprocals, with not more than four scales.

4. Construct nomograms for

(i) $\frac{1}{a^2}+\frac{1}{b^2}=\frac{1}{c^2}$; (ii) $\frac{1}{\sin A}+\frac{1}{\sin B}=\frac{1}{\sin C}$;

(iii) $\frac{ab}{a+b}$; (iv) $\frac{abc}{ab+bc+ca}$.

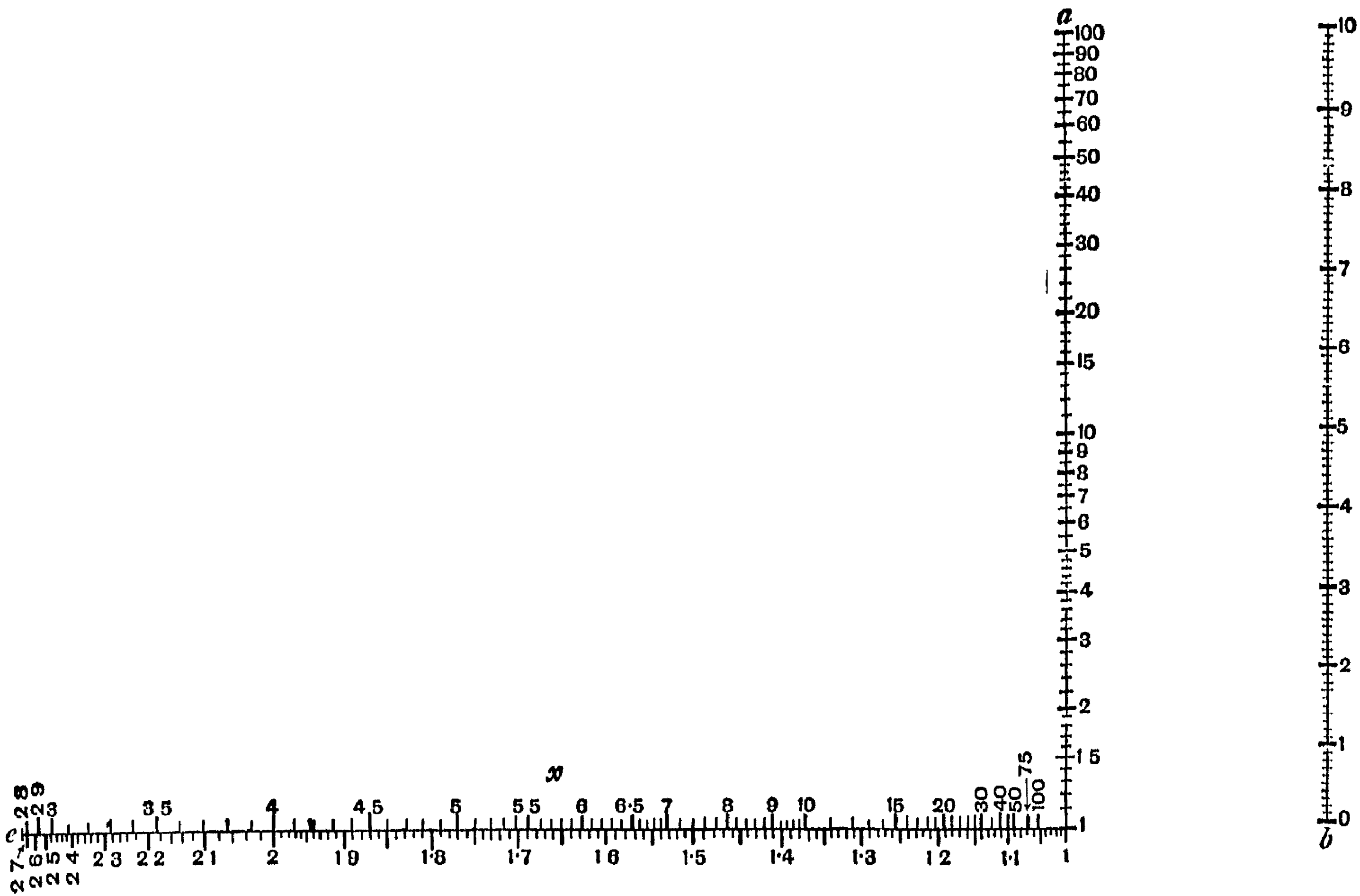


FIG. 64.

CHAPTER VIII

PRACTICAL AND AUTOMATIC CONSTRUCTION OF NOMOGRAMS. EMPIRICAL NOMOGRAMS

72. We have on one or two occasions, after discussing the construction of certain nomograms, had recourse to an automatic process which enables us to make the nomograms without any theoretical discussion. It is, of course, in every way preferable that the student of the subject should have a clear understanding of the principles involved and of the theoretical basis of the construction. Yet, for practical purposes, what is required is the nomogram and not the theoretical work on which it is based. We shall, therefore, devote some space to a more detailed discussion of the practical and automatic method.

73. x = a + b.

Let us return to the simple nomograms of the early chapters of this book. Take, *e.g.*, the nomogram (with parallel a, b scales) for

$$x = a + b.$$

If the point P (Fig. 58), to which is assigned the graduation x, is to be on the x scale of a nomogram for $x=a+b$, then, if we draw two lines A_1PB_1, A_2PB_2 through P, meeting the parallel a, b scales at A_1, A_2, B_1, B_2, we must have

$$\textit{graduation } A_1+\textit{graduation } B_1 = \textit{graduation } A_2+\textit{graduation } B_2,$$

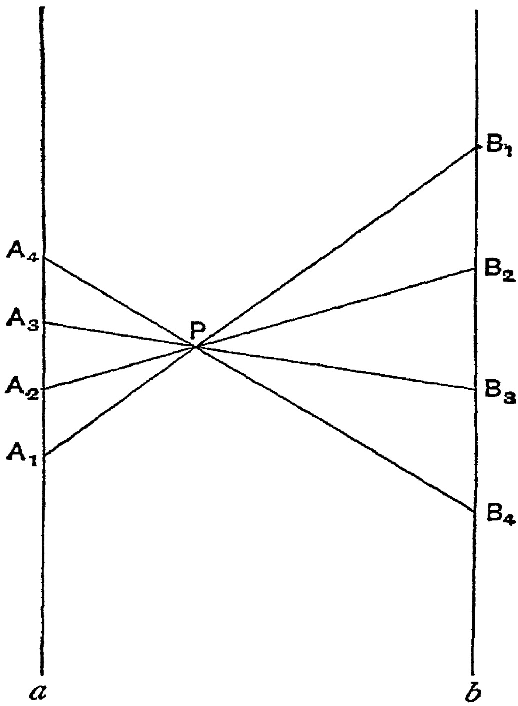


FIG. 58.

or $$\textit{graduation } A_2-\textit{graduation } A_1 = \textit{graduation } B_1-\textit{graduation } B_2.$$

Similarly, if A_3PB_3, A_4PB_4, etc., are other lines through P, we must have

$$\textit{graduation } A_3-\textit{graduation } A_1 = \textit{graduation } B_1-\textit{graduation } B_3,$$

$$\textit{graduation } A_4-\textit{graduation } A_1 = \textit{graduation } B_1-\textit{graduation } B_4, \text{ etc.}$$

It is therefore clear that if the A graduations increase by equal stages $A_1 \rightarrow A_2$, $A_2 \rightarrow A_3$, $A_3 \rightarrow A_4$, etc., then the B graduation must decrease by equal steps, $B_1 \rightarrow B_2$, $B_2 \rightarrow B_3$, $B_3 \rightarrow B_4$, etc. We must therefore graduate the a, b scales uniformly. In our examples in Chapters I., II., we have often adopted equal units in a, b or other units chosen specially. In general this is not necessary. Also the zero points on the two scales can be chosen as we wish, or may find convenient.

Having chosen convenient units in the a, b scales, we can now construct the nomogram as follows. Suppose we wish to find the point P at which the x graduation is to be 5. Since, *e.g.*,

$$5 = 5 + 0,$$

and also $$5 = 0 + 5,$$

we have only to join the 5 on the a scale to the zero on the b scale ; then join the zero on the a scale to the 5 on the b scale. The point of intersection of these lines is the point 5 on the x scale. This is done in Fig. 59 for a few values of x. Although the various x points are obtained in a somewhat erratic manner, it is at once obvious that the x scale is itself a straight line uniformly graduated, and the graduations can now be inserted.

The same method can be employed for the alternative nomogram for addition (§ 19). The reader should argue this out for himself, by taking a figure like Fig. 58, but with the point P outside the space between the a, b scales.

But this method has little to recommend it in the case of addition and subtraction nomograms, and therefore also in the case of multiplication and division

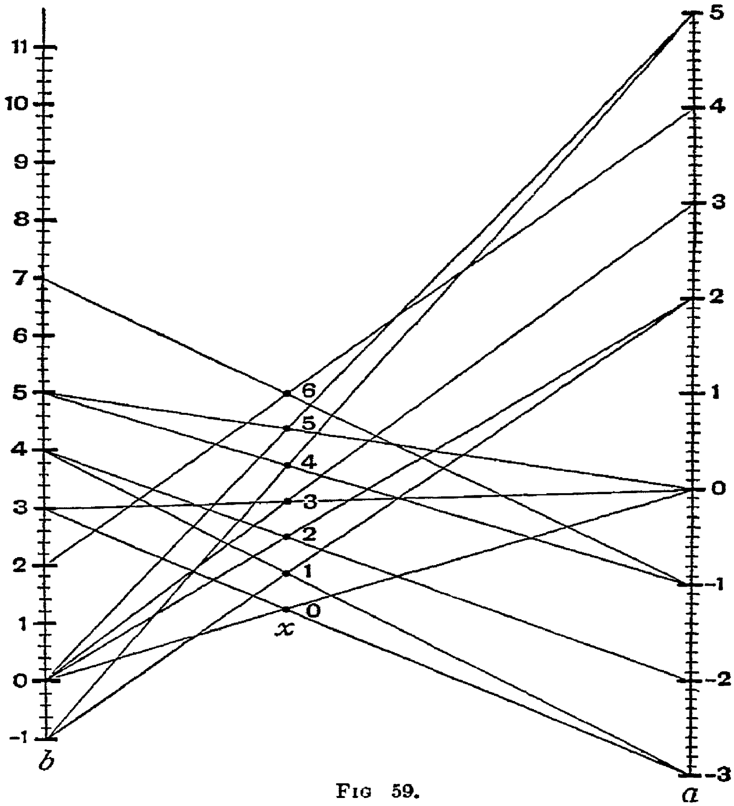


FIG 59.

nomograms, which, as shown in Ch. III., are easily derived from the former. It is an easy matter to construct the nomograms in Ch. III. by the straightforward arguments there given. The scales are all parallel logarithmic scales, and there is little advantage in obscuring the simple algebraic and geometrical ideas underlying the process. The present method, however, is decidedly useful when given ranges are prescribed, see §§ 48-50.

74. $a \sin x + b \cos x = 1$.

When, however, we have to construct more difficult nomograms, as *e.g.* a nomogram for a quadratic equation, this method is really very useful, and often saves much time and labour. We have already illustrated this in the case of the quadratic equation with definite ranges for the a, b scales, and have also indicated its use in the case of the equation

$$a \sin x + b \cos x = 1.$$

The former is discussed in § 49. We shall now give the method again with special reference to the nomogram for

$$a \sin x + b \cos x = 1,$$

discussed theoretically in § 59.

Suppose we wish to construct this nomogram for values of a between 1 and 2, and values of b between 0 and 1. Our object is to find the x curve and its graduations—in other words, we want a succession of points to each of which is assigned a certain graduation x.

We draw the two scales a, b with some convenient units. Since the ranges are equal, we use equal units. The distance apart is chosen so as to make the figure as "square" as possible. This is shown in Fig. 60. Choose some value of b, say $b=0$. Then, if we join this point on the b scale to any point on the a scale, we have somewhere on this line the graduation x given by

$$a = \frac{1}{\sin x}.$$

Thus, to get the x points for

0°, 10°, 20°, 30°, 40°, 50°, 60°, 70°, 80°, 90°

we join the point $b=0$ to the following points, respectively, on the a scale :

∞, 5·76, 2·92, 2, 1·56, 1·31, 1·15, 1·06, 1·02, 1.

Again, take some definite value of a, say $a=2$. Then, if the point $a=2$ is joined to any point b on the b scale,

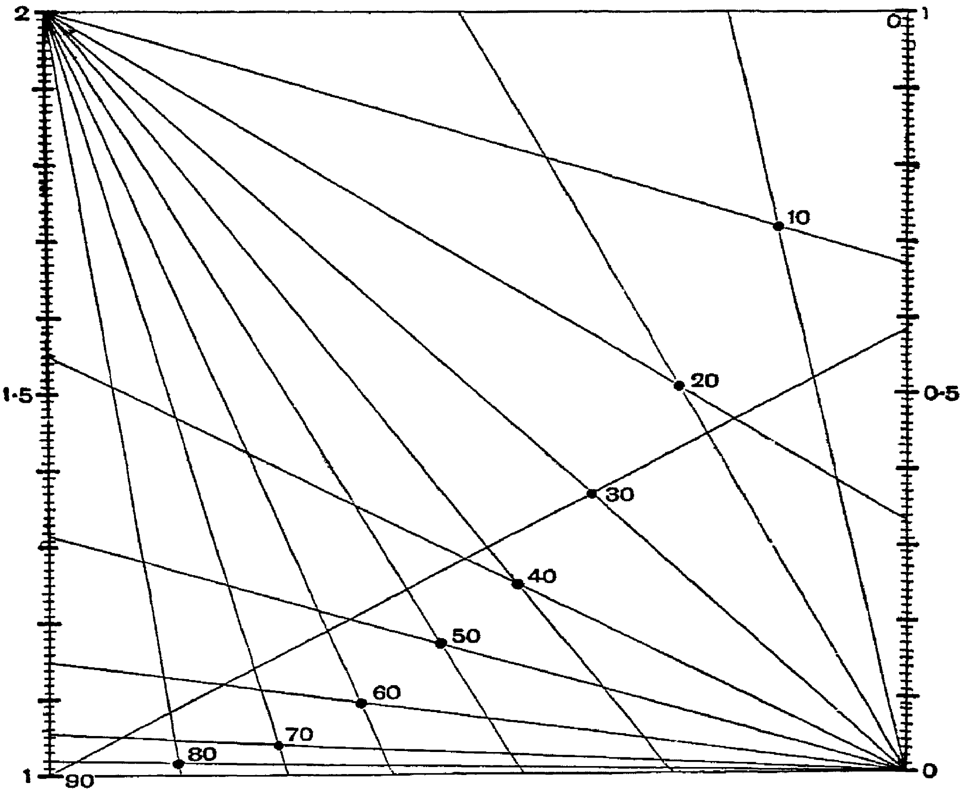


FIG 60.

this join passes through the graduation x on the x curve, if

$$b=\frac{1-2\sin x}{\cos x}=\sec x-2\tan x.$$

Hence to get the x points for

0°, 10°, 20°, 30°, 40°, 50°, 60°, 70°, 80°, 90°, we join the point $a=2$ to the following points, respectively, on the b scale :

$$1,\ 0{\cdot}67,\ 0{\cdot}33,\ 0,\ -0{\cdot}37,\ -0{\cdot}82,\ -1{\cdot}46,\ -2{\cdot}58,\ -5{\cdot}58,\ -\infty.$$

The intersections of the pairs of lines for each value of x gives the points on the x curve for these values of x. In the present instance it happens that the two lines for 30° are really the same. We therefore choose some other means for fixing the position for $x=30°$. *E.g.* we can take $a=1$, and then

$$b=\frac{1-\sin x}{\cos x}$$

$$=1/\sqrt{3}=\sqrt{3}/3=0{\cdot}58.$$

The join of the points $a=1$, $b=0{\cdot}58$. thus gives $x=30°$.

We now join the points thus obtained by as smooth a curve as possible, and insert additional graduations, either by repeating one or other of these processes, or by free-hand interpolation, taking note of the way in which the graduations already obtained suggest these subdivisions. In this way Fig. 60 becomes the nomogram in Fig. 50.

75 (i) $A(x)a+B(x)b+1=0$: Automatic Process

In each case little difficulties may present themselves, and a little ingenuity may be required to overcome them. It is not possible to give a list of such difficulties and the means of overcoming them. This must be left to practice and experience. We shall now give the general

Rule VII. To construct a nomogram for

$$A(x)a + B(x)b + 1 = 0$$

with given ranges for a, b, choose units for a, b, so that these given ranges are as nearly as possible represented by equal lengths, and construct the a, b scales parallel to one another and at a convenient distance apart. Choose some value of b, say b_0; then calculate for a number of values of x the quantity

$$a = -\frac{1 + b_0 B(x)}{A(x)},$$

and join the point b_0 to these points on the a scale. Then choose some value of a, say a_0, and calculate for the same values of x the quantity

$$b = -\frac{1 + a_0 A(x)}{B(x)},$$

and join the point a_0 to these points on the b scale. The intersections of corresponding joins give the points on the x curve for the chosen x values.

Or, instead of choosing b_0 and then a_0, choose two values b_0, b_1, and use the lines given by

$$b = b_0, \quad a = -\frac{1 + b_0 B(x)}{A(x)}; \quad b = b_1, \quad a = -\frac{1 + b_1 B(x)}{A(x)}.$$

or choose two values a_0, a_1, and then use the lines given by

$$a = a_0, \quad b = -\frac{1 + a_0 A(x)}{B(x)}; \quad a = a_1, \quad b = -\frac{1 + a_1 A(x)}{B(x)}.$$

In each case the particular choice to be made must be determined by a few preliminary trials

Thus, in § 74, it is easy to see that if we had chosen a second value of b, say $b = 1$, we should have got unsatisfactory information, as many of these lines are rather close together.

76. $\sin x = a + bx$.

We shall take as a further instance the equation

$$\sin x = a + bx,$$

in which x is to be taken in radians (there are 3·1416 radians in 180°) in the term bx. The ranges for

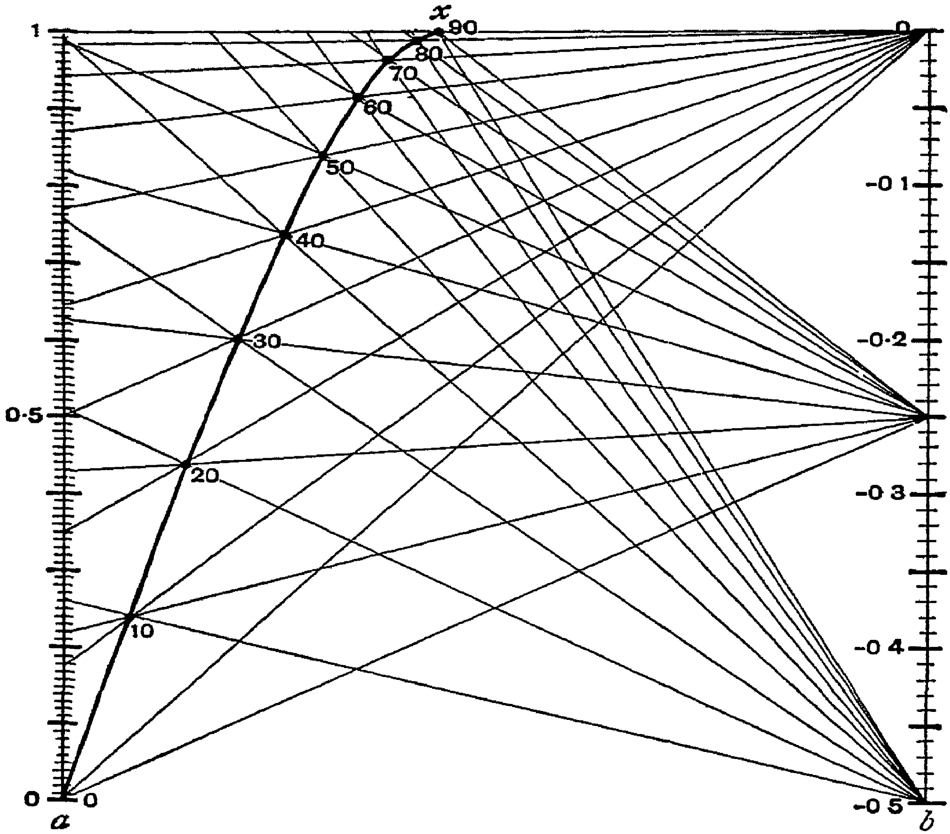


FIG 61.

a, b are supposed to be given as 0 to 1, 0 to $-0·5$ respectively.

We take the b unit twice the a unit and construct the a, b scales as in Fig. 61. After a little trial we find that it is convenient to choose $b = 0$ and $b = -0·5$.

For the former we have $a = \sin x$, for the latter we have $a = \sin x + \frac{1}{2}x$. If we wish the nomogram to read in degrees, we take

$$x = 0°,\ 10°,\ 20°,\ 30°,\ 40°,\ 50°,\ 60°,\ 70°,\ 80°,\ 90°.$$

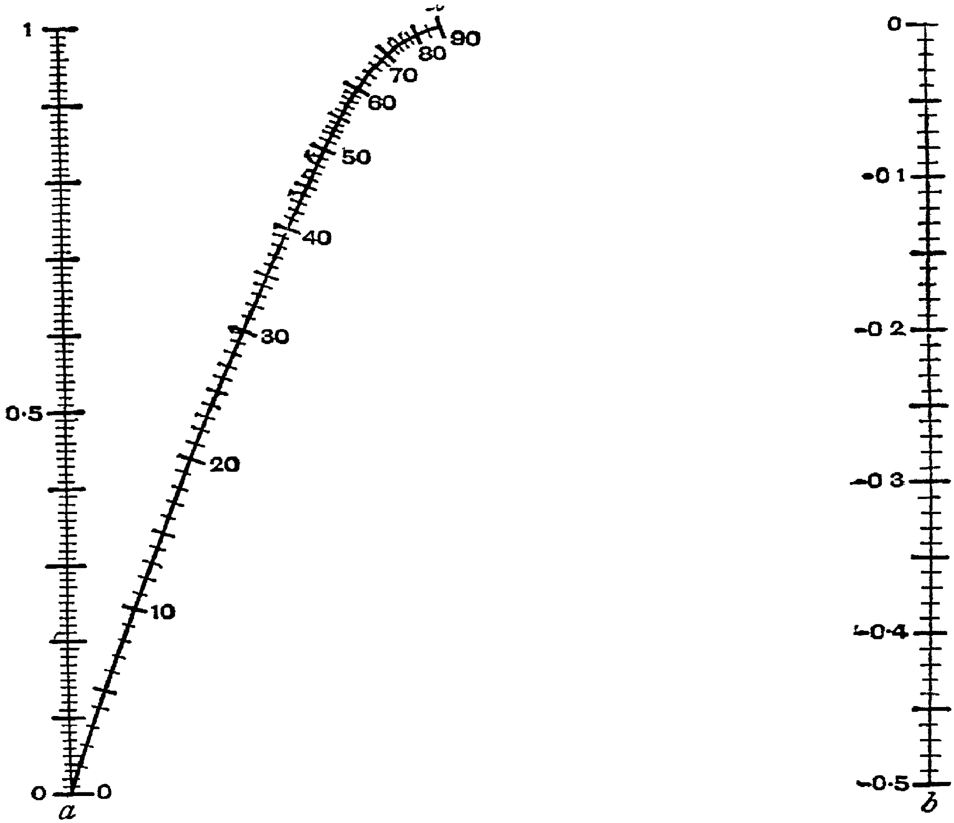


FIG 62.

This means that we have to join the point $b = 0$ to the a points

0, 0·17, 0·34, 0·50, 0·64, 0·77, 0·87, 0·94, 0·99, 1·0 ;

and the point $b = -0{\cdot}5$ to the a points

0, 0·26, 0·52, 0·76, 0·99, 1·20, 1·39, 1·55, 1 68, 1·79.

The respective intersections give the x points for these values of x, and it is clear that no values of x are needed outside this range. We put in the useful subdivisions, and we get the nomogram in Fig. 62.

77. In carrying out the practical process of this chapter it is evidently important to have some check or test for the accuracy of the result. The first useful check is the fact that the isolated x points thus obtained must lie on a *smooth* curve, and that the intervals must grow or diminish in some continuous manner and not erratically. The next check consists in actually testing the resulting nomogram. Thus, let us take $b=-0{\cdot}25$. Then we have $a=\sin x+\frac{1}{4}x$. If then we join the point $b=-0{\cdot}25$ to the a points

0, 0·22, 0·43, 0·63, 0·82, 0·98, 1·13, 1·25, 1·33, 1·39,

the lines must pass through the x points found. This condition is seen to be satisfied in Fig. 61.

78 **Empirical Nomograms.**

The method of this chapter is particularly valuable in the case of relations which are only obtained *empirically*, *i.e.* from a number of experiments, and for which the algebraic equation is not known. If we know that the relation between x, a, b is of the form

$$A(x)a+B(x)b+1=0,$$

but do not know the forms of the functions $A(x)$, $B(x)$, we can still construct the nomogram by Rule VII, § 75. All that is needed is the information mentioned in the rule. Thus, if for one value of a, say a_0, we know the values of b for a number of values of x, and for another value of a, say a_1, we know the values of b for the same values of x, we can draw two sets of lines as suggested in the rule, and the intersections of corresponding lines will be the x points on the x curve.

If we are not sure that the relation

$$A(x)a+B(x)b+1=0$$

is a correct representation of the facts, this is tested by the process; for on drawing the lines for some other value of a, say a_2, we at once see whether the suggested relation is correct by the accuracy with which the new set of lines pass through the x points already found. Great certainty can be attained by the application of this test several times with sets of lines through a few chosen values of a or b.

79. (ii) $a^{A(x)}b^{B(x)}=C(x)$: Automatic Process.

We have so far taken the case where the relation between x, a, b is of the form

$$A(x)a+B(x)b+1=0,$$

or, more generally,

$$A(x)a+B(x)b+C(x)=0,$$

for which, in accordance with the analysis of § 52, we use parallel coordinates. If the relation is of the form

$$a^{A(x)}b^{B(x)}=C(x),$$

such as, *e.g.*, $x=ab$, $x^2=a^3b^4$, etc., we take logarithms of both sides, and we deduce a relation of the form already used. Thus, for this new type of relation we take *logarithmic* scales for a, b and proceed in the same way.

80. $bx = a^x$.

As an instance take the equation

$$x = a^x / b,$$

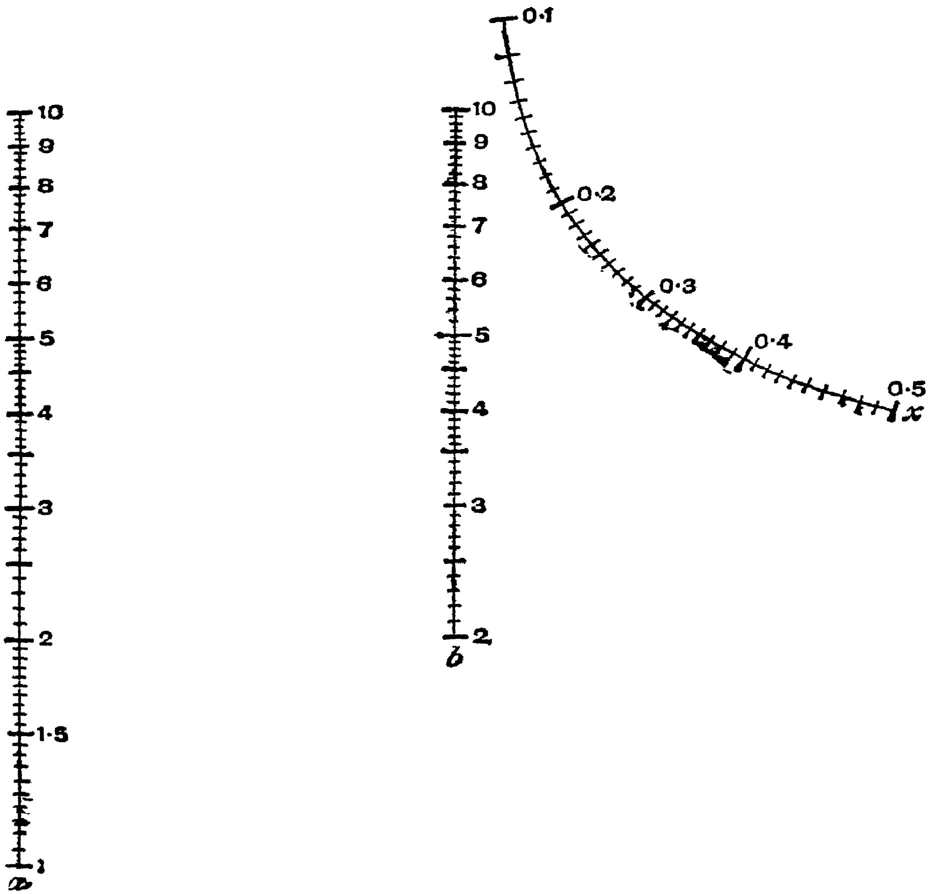


FIG 63.

in which a ranges between 1 and 10, and b between 2 and 10. We get, by taking logarithms,

$$\log x = x \log a - \log b,$$

and this is of the form already discussed. We take two logarithmic scales with the same unit for a, b (Fig. 63). If we choose $a=1$, we get

$$b=\frac{1}{x},$$

giving a set of lines through the point $a=1$. If we choose, in addition, $a=10$, we get a set of lines defined by this point and

$$b=\frac{10^x}{x}.$$

The intersections of these lines give the x curve and the graduations as before. The result can be tested as suggested in § 77.

81. (iii) $a^{A(x)}[B(x)]^b=C(x)$: Automatic Process.

There is another type of equation that can be treated in the same way; this is one in which a, say, occurs in the form log a, and b as an ordinary number, so that, if log a, b are parallel coordinates, the equation is of the form

$$A(x)\log a+B(x)b+C(x)=0.$$

It is readily seen that any equation of the type

$$C(x)=a^{A(x)}[B(x)]^b$$

belongs to this form.

82. $a^x=x^b$.

An example of this sort is given by the equation

$$a^x=x^b.$$

Taking logarithms, we get

$$x\log a=b\log x.$$

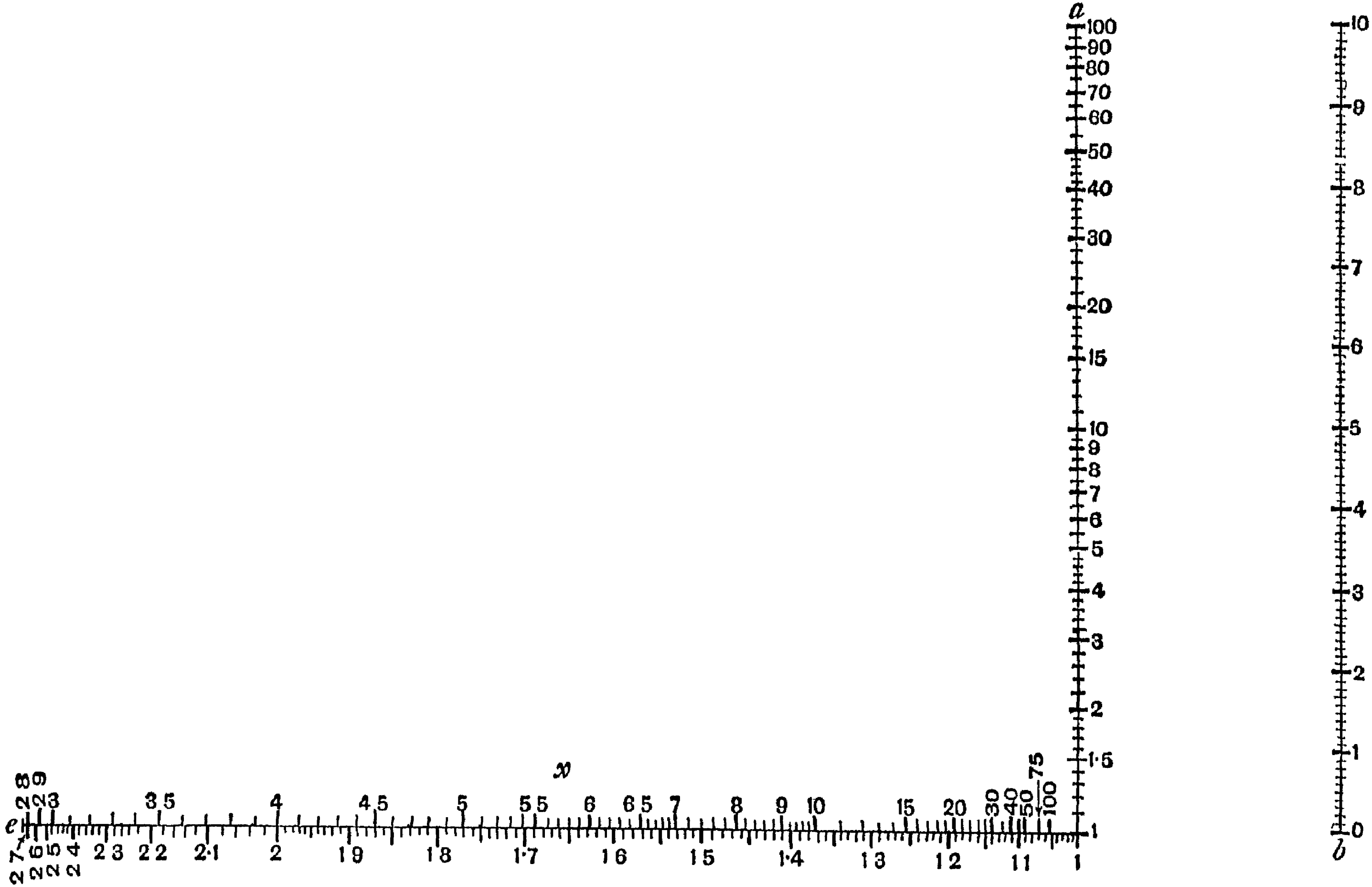

a
100
90
80
70
60
50
40
30
20
15
10
9
8
7
6
5
4
3
2
1·5
1
b
10
9
8
7
6
5
4
3
2
1
0
x
e

FIG. 64.

It will be found convenient to take the unit of $\log a$ 5 times the b unit, and construct a uniform b scale and a logarithmic a scale, as in Fig. 64.

To get points on the x curve we search for means of getting sets of lines on which certain values of x will lie. One idea suggests itself at once. The equation is clearly satisfied for $a=b=x$, no matter what this value may be. Hence the line joining $a=1$ to $b=1$ must contain the point 1 on the x curve; the line joining $a=2$ to $b=2$ must contain the point 2 on the x curve; and so on. This device is, of course, special to this equation, but it is given here as an example of the sort of thing of which one must be ready to take advantage.

Again, if $a=1$, we must have $b=0$ for all values of x. Hence we let the lines already obtained cut the line joining $a=1$ to $b=0$, and the points of intersection are the corresponding x points required. The x curve happens to be a straight line. Note that on the x scale each point has two graduations, but that only part of the x line is graduated at all. This indicates that it is not always possible to find values of x to satisfy the equation for given a, b. It is curious that the extreme graduation is e, the theoretical quantity 2·71828.... The mathematical reader will find it interesting to investigate this point.

In general, equations of this third type will have to be treated in the same manner as the first two types, (i), (ii).

83. Double Graduations.

The double "graduations" that arise in Fig. 64 can be imitated artificially in problems where different units may be used by different people. Thus, if we make a nomogram for $s=ut$, giving the distance described in time t, when the velocity is u, we may on one side, say the left, of the u scale, graduate it in ft./sec., on the other side in miles/hour; the t scale can be graduated in seconds on the left, hours on the right; and the s scale in feet on the left and miles on the right. In using this nomogram, if attention is paid to the graduations on the left, we get the answer when the units are feet and seconds; if attention is paid to the graduations on the right, we get the answer when the units are miles and hours. The student will have no difficulty in applying this device.

(Do, by the automatic method, some of the examples at the ends of Chapters III., IV., V., VI., using suitable ranges.)

EXAMPLES VIII.

1. Construct nomograms for

(i) $\sin x = a + b\cos x$, a, b ranging 0 to 1, 0 to $-\frac{1}{2}$;
(ii) $a\log_{10} x = x + b$, a, b ranging 0 to 1, 1 to 10;
(iii) $a10^x + b10^{-x} = 1$, a, b ranging 0 to $\frac{1}{2}$;
(iv) $\log(1+x) = ax + bx^2$, a, b ranging $+\frac{1}{10}$ to $-\frac{1}{10}$.

2. Construct nomograms with suitable ranges for

(i) $bx^2 = a^x$; (ii) $a^x b^{x^2} = 1$; (iii) $a^x b^{x^2} = x^3$;
(iv) $bx^2 = a^{\frac{1}{x}}$; (v) $a^x b^{\frac{1}{x}} = 1$; (vi) $a^x b^{\frac{1}{x}} = 2$.
(vii) $a^x b = x^2$; (viii) $a^x b^{\frac{1}{x}} = 3x^2$; (ix) $a^x b^{-\frac{1}{x}} = 2x$.

3. Construct nomograms with suitable ranges for

(i) $a^x = 2x^b$; (ii) $a^{2x} = x^{3b}$; (iii) $a^{x+1} = x^b$;

(iv) $a^x x^b = x$; (v) $a^{2x} x^{2b} = x$; (vi) $a^{x'} x^b = \frac{1}{x}$.

4. Construct doubly graduated nomograms for

(i) $v = ft$ (feet and seconds, centimetres and seconds);

(ii) K.E. $= \frac{W}{2g} v^2$ (miles/hour and lbs, cm./sec. and kilograms);

(iii) $R = Kr^2U^2$ (r in feet and U in ft./sec., r in cm. and U in cm./sec);

(iv) $h = \frac{v^2}{2g}$ (v in ft./sec. and f in ft./sec./sec., v in cm./sec. and f in cm/sec./sec).

5. Construct a nomogram for $x = \frac{a - b}{\log (a, b)}$.

6. Construct a nomogram for $H = \frac{H_1}{x} + H_2 x^3$ for H_1, H_2 between 10 and 50, and x between $\frac{1}{2}$ and **3**.

CHAPTER IX

METHOD OF DETERMINANTS. FOUR VARIABLES

84. Determinants of the Second Order.

For the purpose of the present chapter the reader will need a certain acquaintance with the theory of Determinants. This is best obtained by reference to some standard work on Algebra, but the information indispensable for application to nomography is comparatively simple.

Suppose we have *four* quantities or *elements* :

$$a_1, b_1 ;$$
$$a_2, b_2.$$

The expression of *two terms*

$$a_1b_2 - a_2b_1$$

occurs frequently in algebraic applications. Thus, if we have the simultaneous equation

$$a_1x + b_1y = 1,$$
$$a_2x + b_2y = 1,$$

then the solutions are

$$x = \frac{b_2 - b_1}{a_1b_2 - a_2b_1}, \quad y = \frac{a_1 - a_2}{a_1b_2 - a_2b_1}$$

The denominators are the same in both, and is in fact the expression just mentioned.

Let us introduce for $a_1b_2-a_2b_1$ the notation

$$\begin{vmatrix} a_1, & b_1 \\ a_2, & b_2 \end{vmatrix};$$

we call this new expression a *determinant* of *two rows and columns*, or of the *second order*. The rows are characterised by the suffixes 1, 2 respectively, while the columns are characterised by the letters a, b respectively. The value of this notation is at once grasped by considering the solutions of the equations

$$a_1x+b_1y=c_1,$$
$$a_2x+b_2y=c_2.$$

These solutions are

$$x=\frac{c_1b_2-c_2b_1}{a_1b_2-a_2b_1},\quad y=\frac{a_1c_2-a_2c_1}{a_1b_2-a_2b_1};$$

which with the determinantal notation can be written

$$x=\frac{\begin{vmatrix} c_1, & b_1 \\ c_2, & b_2 \end{vmatrix}}{\begin{vmatrix} a_1, & b_1 \\ a_2, & b_2 \end{vmatrix}},\qquad y=\frac{\begin{vmatrix} a_1, & c_1 \\ a_2, & c_2 \end{vmatrix}}{\begin{vmatrix} a_1, & b_1 \\ a_2, & b_2 \end{vmatrix}}.$$

It is clear that this notation gives simple forms to the solutions, forms that are easily memorised.

If we multiply the two elements of either column by the same quantity, k, so that the determinant is now

$$\begin{vmatrix} ka_1, & b_1 \\ ka_2, & b_2 \end{vmatrix} \quad\text{or}\quad \begin{vmatrix} a_1, & kb_1 \\ a_2, & kb_2 \end{vmatrix},$$

the value of the determinant is also multiplied by k. If we multiply the elements of either row by any such quantity, then once again the determinant is multiplied by this quantity. These statements are obvious by definition.

It is equally obvious that if we interchange the two columns, the determinant retains its numerical value but has its sign changed. If we interchange the two rows, the determinant is again changed in sign, but is unchanged in numerical value.

Again, if we multiply the elements of a column by the same quantity, k, and add to the corresponding elements of the other column, the determinant is unchanged in value. For example, let us multiply the elements of the first column by k, and add to the corresponding elements of the second column. The new determinant is

$$\begin{vmatrix} a_1, & b_1+ka_1 \\ a_2, & b_2+ka_2 \end{vmatrix},$$

and this is equal to

$$a_1(b_2+ka_2)-a_2(b_1+ka_1),$$

which reduces to $\quad a_1b_2-a_2b_1.$

In the same way, if the elements of one of the rows are multiplied by the same quantity, and added to the corresponding elements of the other row, the determinant is unchanged in value.

It now follows that *if such a determinant is equal to zero*, the equation remains true, *i.e.* the determinant is still equal to zero, if we carry out any of the processes thus enumerated. We can multiply the elements of any column or of any row by a given constant; we can add to the elements of any row or column the corresponding elements of the other row or column multiplied by any given quantity; we can interchange the two rows or the two columns: any of these processes, or *any number of such processes, carried out in any order, do not affect the equation, and the determinant obtained is still zero.*

85. Determinants of the Third Order.

Let us now extend the definition as follows. Suppose we have *nine* quantities or *elements* in three sets of three each, namely :

$$\begin{aligned} &a_1, \quad b_1, \quad c_1\,; \\ &a_2, \quad b_2, \quad c_2\,; \\ &a_3, \quad b_3, \quad c_3. \end{aligned}$$

The expression of *six terms*

$$a_1b_2c_3 - a_1b_3c_2 + a_2b_3c_1 - a_2b_1c_3 + a_3b_1c_2 - a_3b_2c_1$$

is of frequent occurrence. The reader will easily verify for himself that the solutions of the three simultaneous equations

$$\begin{aligned} a_1x + b_1y + c_1z &= 1, \\ a_2x + b_2y + c_2z &= 1, \\ a_3x + b_3y + c_3z &= 1, \end{aligned}$$

all contain this expression in the denominators. It is therefore convenient to have a notation for this six-termed expression, and the notation used is

$$\begin{vmatrix} a_1, & b_1, & c_1 \\ a_2, & b_2, & c_2 \\ a_3, & b_3, & c_3 \end{vmatrix},$$

a *determinant* of *three rows and columns*, or of the *third order*.

Just as in the case of the determinant with two rows and columns, if we multiply the elements of any row by a given number k, the whole determinant is multiplied by this number. This is because if, say, we multiply the elements of the first row by k, then in the expression

$$a_1b_2c_3-a_1b_3c_2+a_2b_3c_1-a_2b_1c_3+a_3b_1c_2-a_3b_2c_1$$

all the elements with the suffix 1 are multiplied by k, and each term contains one such element as a factor. In the same way, if the elements of, say, the first column are multiplied by a given number, the determinant is multiplied by this number, because, in this same expression, each element a is multiplied by the same number, and each term in the expression contains one such element as a factor.

Suppose now that we interchange two of the rows, for example, the second and third. The value of the new determinant is obtained by interchanging the suffixes 2 and 3, so that we get

$$a_1b_3c_2-a_1b_2c_3+a_3b_2c_1-a_3b_1c_2+a_2b_1c_3-a_2b_3c_1,$$

and this is the original expression with all the signs reversed. The determinant thus retains its numerical value, and only its sign is changed. The same applies to any other such interchange of rows, and also to any interchange of columns.

Finally, if to the elements of any row we add the corresponding elements of any other row multiplied by a given quantity, the value of the determinant is unaltered. For example, let k times the second row be added to the first, so that the determinant is now

$$\begin{vmatrix} a_1+ka_2, & b_1+kb_2, & c_1+kc_2 \\ a_2, & b_2, & c_2 \\ a_3, & b_3, & c_3 \end{vmatrix}.$$

When this is worked out we get

$$(a_1+ka_2)(b_2c_3-b_3c_2)+a_2b_3(c_1+kc_2)-a_2(b_1+kb_2)c_3$$
$$+a_3(b_1+kb_2)c_2-a_3b_2(c_1+kc_2),$$

and this reduces to the original value. The same applies to columns: if we add to the elements of any one column a given number of times the corresponding elements of any other column, the value of the determinant is unchanged.

Suppose now that the third order determinant is equated to zero. It follows that the determinant is still equal to zero if we multiply any row or any column by a given number; if we interchange any two rows or any two columns; or if we add to any row any multiple of either of the other rows or to any column any multiple of either of the other two columns.

In dealing with determinants of the third order one case of particular importance is that in which all three elements of the third column are unity. This means

$$c_1=c_2=c_3=1.$$

The determinant is now

$$\begin{vmatrix} a_1, & b_1, & 1 \\ a_2, & b_2, & 1 \\ a_3, & b_3, & 1 \end{vmatrix},$$

which when worked out becomes

$$a_1b_2-a_1b_3+a_2b_3-a_2b_1+a_3b_1-a_3b_2.$$

It is this particular type of determinant of the third order that we shall apply to nomography.

86. Collinear Points: Determinantal Condition

The method of nomography is based upon the properties of three collinear points. Now let (ξ_1, η_1),

(ξ_1, η_2), (ξ_3, η_3) be the cartesian coordinates of three collinear points; then there must be a relationship which expresses the condition of collinearity. It is that the gradient of the line joining the first point to the second point must be the same as the gradient of the line joining the first point to the third point. This means

$$\frac{\eta_2 - \eta_1}{\xi_2 - \xi_1} = \frac{\eta_3 - \eta_1}{\xi_3 - \xi_1},$$

which, when worked out, gives

$$\xi_1\eta_2 - \xi_1\eta_3 + \xi_2\eta_3 - \xi_2\eta_1 + \xi_3\eta_1 - \xi_3\eta_2 = 0.$$

This relationship must be satisfied if the three points are collinear. Using a determinant with three rows and columns, we can write this condition in the form

$$\begin{vmatrix} \xi_1, & \eta_1, & 1 \\ \xi_2, & \eta_2, & 1 \\ \xi_3, & \eta_3, & 1 \end{vmatrix} = 0.$$

87. Derivation of Nomograms : $x^2 + ax + b = 0$.

Let us suppose that ξ_1, η_1 are both functions of a variable x; this means that the first point lies on a curve defined by giving the coordinates (ξ_1, η_1) of any point on it in terms of the parameter x. In the same way let ξ_2, η_2 be functions of a variable a, so that the second point lies on a curve defined by giving the coordinates (ξ_2, η_2) of any point on it in terms of the parameter a. Further, let ξ_3, η_3 be functions of a variable b, so that the third point lies on a curve defined by giving the coordinates (ξ_3, η_3) of any point on it in terms of the parameter b. The three curves can be plotted and graduated in terms of x, a, b.

For example, suppose that

$$\xi_1 = -\frac{x}{x+1}, \quad \eta_1 = -\frac{x^2}{x+1};$$

$$\xi_2 = -1, \quad \eta_2 = a;$$

$$\xi_3 = 0, \quad \eta_3 = b.$$

It is seen that the second point lies on the straight line $\xi = -1$. The third point lies on the straight line $\xi = 0$. The first point lies on the curve

$$\eta = -\frac{\xi^2}{\xi+1}.$$

This is an hyperbola with an asymptote along the line $\xi = -1$. (See page 78, Rule VI.)

Now the parameters x, a, b of the three points in which the three curves thus defined are cut by any straight line must satisfy the determinantal equation of § 86. When ξ_1, η_1 are given in terms of x, ξ_2, η_2 in terms of a, and ξ_3, η_3 in terms of b, we have in fact an equation for x appropriate to the particular straight line in question, and involving a, b. Thus, in the example just taken, the equation connecting the three graduations x, a, b in which the hyperbola and the a, b scales are cut by any straight line is

$$\begin{vmatrix} -\frac{x}{x+1}, & -\frac{x^2}{x+1}, & 1 \\ -1, & a, & 1 \\ 0, & b, & 1 \end{vmatrix} = 0,$$

which readily reduces to

$$x^2 + ax + b = 0.$$

Hence we see that the solution of the quadratic equation

$$x^2+ax+b=0$$

is obtained by means of the nomogram defined in Chapter IV., § 47, as d'Ocagne's Nomogram.

We can now see how this method works more generally.

Rule VIII. **Let ξ_1, η_1 involve only x, ξ_2, η_2, involve only a, and ξ_3, η_3, involve only b; so that (ξ_1, η_1) lies on a curve whose graduations are in terms of x, (ξ_2, η_2) lies on a curve whose graduations are in terms of a, (ξ_3, η_3) lies on a curve whose graduations are in terms of b. Then three collinear points on these three curves give a value of x corresponding to values of a, b as a solution of the equation.**

$$\begin{vmatrix} \xi_1, & \eta_1, & 1 \\ \xi_2, & \eta_2, & 1 \\ \xi_3, & \eta_3, & 1 \end{vmatrix}=0$$

in which for ξ_1, η_1; ξ_2, η_2; ξ_3, η_3 we substitute their values in terms of x, a, b respectively.

88. Further Examples.

As a further example use the same values of ξ_2, η_2; ξ_3, η_3 as before, but now let

$$\xi_1=-\frac{x}{x+1}, \quad \eta_1=-\frac{x^3}{x+1}.$$

The reader will easily verify that we get a nomogram for

$$x^3+ax+b=0.$$

Again, suppose that

$$\xi_1=\frac{x-1}{x+1}, \quad \eta_1=-\frac{x^3}{x+1};$$

$$\xi_2=1, \qquad \eta_2=a;$$

$$\xi_3=-1, \qquad \eta_3=b.$$

We see at once that the equation for x in terms of a, b is

$$\begin{vmatrix} \frac{x-1}{x+1}, & -\frac{x^3}{x+1}, & 1 \\ 1, & a, & 1 \\ -1, & b, & 1 \end{vmatrix} = 0,$$

which reduces to

$$x^3 + ax + b = 0.$$

We thus have another nomogram for the cubic equation, but now the a scale is along the line $\xi = 1$, and the b scale is along the line $\xi = -1$.

Whittaker's nomogram (see § 55) for the quadratic equation can be obtained by this method as follows. Let

$$\xi_1 = \frac{1}{1+x^2}, \quad \eta_1 = \frac{x}{1+x^2};$$

$$\xi_2 = \frac{1}{1-b}, \quad \eta_2 = 0;$$

$$\xi_3 = 0, \quad \eta_3 = -\frac{1}{a}.$$

The second point lies on the line $\eta = 0$. The third point lies on the line $\xi = 0$. The first point lies on the circle $\xi^2 + \eta^2 = \xi$. Also the equation between x, a, b is

$$\begin{vmatrix} \frac{1}{1+x^2}, & \frac{x}{1+x^2}, & 1 \\ \frac{1}{1-b}, & 0, & 1 \\ 0, & -\frac{1}{a}, & 1 \end{vmatrix} = 0,$$

which becomes

$$x^2+ax+b=0.$$

We have in fact the circular nomogram of Fig. 48.

89. Different Nomograms for same Equation.

The method of determinants can also be used for the nomograms of Chapter III., but it is clear that the application of the method in such cases is really a waste of labour. The immediate value of the determinantal method lies in other considerations.

We have seen that if in a determinant the elements of any one column are each multiplied by the same quantity, and added to the corresponding elements of another column, then the value of the determinant is unaltered.

If now in the determinantal equation of § 86, where in the first row only x occurs, in the second row only a, in the third row only b, we add to the elements of any column some multiple of the elements of another column, we get a new determinantal equation of the same type as before. But the value of the determinant is unaltered. Hence we have another nomogram for the same equation, and so it is possible to study the different ways of dealing with any one equation.

For example, take the quadratic equation

$$x^2+ax+b=0,$$

as dealt with in § 87. In the determinant

$$\begin{vmatrix} -\dfrac{x}{x+1}, & -\dfrac{x^2}{x+1}, & 1 \\ -1, & a, & 1 \\ 0. & b. & 1 \end{vmatrix}=0$$

add the first column to the second. We now get the determinantal equation

$$\begin{vmatrix} -\frac{x}{x+1}, & -x, & 1 \\ -1, & a-1, & 1 \\ 0, & b, & 1 \end{vmatrix} = 0.$$

This means that to solve the equation

$$x^2+ax+b=0$$

we can use collinear points

$$\xi_1 = -\frac{x}{x+1}, \quad \eta_1 = -x\,;$$
$$\xi_2 = -1, \quad \eta_2 = a-1\,;$$
$$\xi_3 = 0, \quad \eta_3 = b.$$

We have the same a, b scales as before, except that the a scale is now graduated differently, while the x curve now has the equation

$$\xi\eta - \xi + \eta = 0.$$

It is somewhat easier to deal with the x curve in this form, since both asymptotes are parallel to the coordinates axes.

Other manipulations are also possible. Thus, take the same determinantal equation

$$\begin{vmatrix} -\frac{x}{x+1}, & -\frac{x^2}{x+1}, & 1 \\ -1, & a, & 1 \\ 0, & b, & 1 \end{vmatrix} = 0,$$

and divide the first row by $-x^2/(x+1)$, the second row by a, the third row by b; interchange the second and third columns, and finally add the first column to the new second column. We get the determinantal equation

$$\begin{vmatrix} \frac{1}{x}, & -\frac{1}{x^2}, & 1 \\ -\frac{1}{a} & 0, & 1 \\ 0, & \frac{1}{b}, & 1 \end{vmatrix} = 0.$$

This means that we can use

$$\xi_1 = \frac{1}{x}, \qquad \eta_1 = -\frac{1}{x^2};$$

$$\xi_2 = -\frac{1}{a}, \qquad \eta_2 = 0;$$

$$\xi_3 = 0, \qquad \eta_3 = \frac{1}{b}.$$

We have in fact the parabolic nomogram of Fig. 38.

It is clear that in this way we can examine different nomograms for any given equation, and decide upon the most convenient form for any particular purpose.

90. General Type.

The general type of equation to which the determinantal method is immediately applicable can be deduced from

$$\begin{vmatrix} \xi_1, & \eta_1, & 1 \\ \xi_2, & \eta_2, & 1 \\ \xi_3, & \eta_3, & 1 \end{vmatrix} = 0.$$

In this equation ξ_1, η_1 are functions of x, ξ_2, η_2 are functions of a, and ξ_3, η_3 are functions of b. Writing

$$X(x), \quad X'(x)$$

for ξ_1, η_1,

$$A(a), \quad A'(a)$$

for ξ_2, η_2,

$$B(b), \quad B'(b)$$

for ξ_3, η_3, we get the equation

$$X(x)\{A'(a)-B'(b)\}-X'(x)\{A(a)-B(b)\} \\ +\{A(a)B'(b)-A'(a)B(b)\}=0$$

as the general fundamental type of equation soluble by the method of determinants. We see that the equation of the type given in § 52, to which parallel coordinates are immediately applicable, is a special case of this general type, in which $A(a)$ and $B'(b)$ are constants or zero.

91. Equations with Four Quantities. Family of x Curves.

The method of determinants suggests immediately a very important means of dealing with equations involving four variable quantities. It is true that we have already examined problems with four or more quantities in Chapter III. The method there, however, is one based essentially upon the *successive* application of the three variable method, and is not usable if the four or more variables cannot be arranged in such a manner. Thus suppose we have the equation

$$x^3+\lambda x^2+ax+b=0$$

in which the coefficients λ, a, b can assume any values (possibly restricted to lie within certain limits). In this equation we cannot deal with x, a, b first and then introduce λ, as we would do if we wished to make a nomogram for, say,

$$X=LAB.$$

This is because in the cubic equation just mentioned all four variables must be dealt with *simultaneously*.

Let us use the determinantal equation of § 86 in which we make

$$\xi_1 = -\frac{x}{x+1}, \quad \eta_1 = -\frac{x^3+\lambda x^2}{x+1};$$

$$\xi_2 = -1, \quad \eta_2 = a;$$

$$\xi_3 = 0, \quad \eta_3 = b.$$

The equation is now

$$\begin{vmatrix} -\frac{x}{x+1}, & -\frac{x^3+\lambda x^2}{x+1}, & 1 \\ -1, & a, & 1 \\ 0, & b, & 1 \end{vmatrix} = 0,$$

which works out to be

$$x^3+\lambda x^2+ax+b=0.$$

The a, b scales are the same as in § 87, but the defintions

$$\xi_1 = -\frac{x}{x+1}, \quad \eta_1 = -\frac{x^3+\lambda x^2}{x+1},$$

mean that the point (ξ_1, η_1) lies on one member of a *family of curves*, depending upon the particular value of λ. This suggests that in order to solve the cubic equation in which all the terms are present, it is necessary to draw not one x curve, but a *family of x curves*, each curve corresponding to some particular value of the coefficient of x^2.

In Fig. 65 we have drawn these curves for $\lambda = -10$ to $\lambda = 10$.

In order to facilitate the use of a nomogram containing a family of x curves, it is advantageous to join up the points on the different x curves corresponding to any given value of x. Thus we find in Fig. 65 that

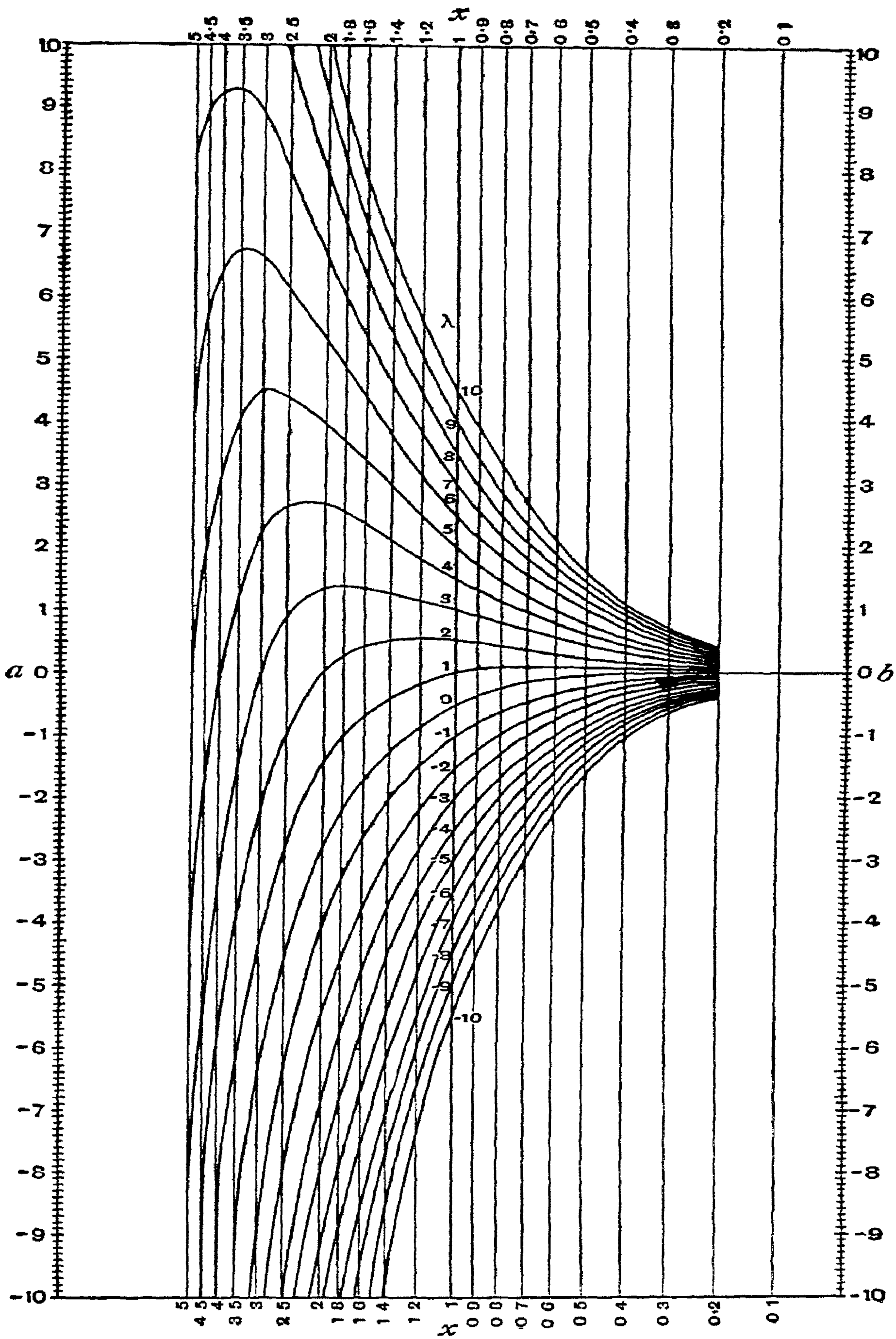

x
λ
a
b
10
9
8
7
6
5
4
3
2
1
0
-1
-2
-3
-4
-5
-6
-7
-8
-9
-10
5
4·5
4
3·5
3
2·5
2
1·8
1·6
1·4
1·2
1
0·9
0·8
0·7
0·6
0·5
0·4
0·3
0·2
0·1

FIG 65.

all the x graduations for any given value of x, and for *all* values of λ, lie on the straight line

$$\xi = -\frac{x}{x+1}.$$

By drawing all these straight lines we get *another family of curves*. The x curves originally drawn correspond to constant values of λ, and the new curves correspond to constant values of x. We thus have a network consisting of *two intersecting families*, and any point in the network corresponds to definite values of x and λ.

This applies generally to any equation for which we can use the method here discussed. This is when ξ_2, η_2 are functions of a, ξ_3, η_3 are functions of b, while the definitions of ξ_1, η_1 in terms of x and λ suggest that any point (ξ_1, η_1) can be considered as lying on two curves, one being obtained by eliminating x between ξ_1, η_1 so that we have a curve for a given value of λ, and the other by eliminating λ between ξ_1, η_1, so that we have a curve for a given value of x.

The general type of equation to which this method is applicable is

$$X(x, \lambda)\{A'(a)-B'(b)\} - X'(x, \lambda)\{A'(a)-B(b)\} \\ +\{A(a)\,B'(b)-A'(a)\,B(b)\}=0,$$

in which either X or X' or both X and X' are functions of x and λ.

92. Nomogram for Mixtures.

An interesting example of a nomogram with four variables is the nomogram for mixtures. Suppose that three substances of given prices are to be mixed so as to produce a mixture of a given price. It is clear that there are an infinite number of ways of doing this: we can find them as follows.

Let the mixture consist of

a % of substance 1 whose price is α,

b % of substance 2 whose price is β,

c % of substance 3 whose price is γ,

α, β, γ being in *descending* order of magnitude. Let the required price of the mixture be δ. We have two equations for a, b, c, namely,

$$a+b+c=100,$$

$$a\alpha+b\beta+c\gamma=(a+b+c)\,\delta.$$

The second equation can be written

$$a\,(\alpha-\delta)+b\,(\beta-\delta)+c\,(\gamma-\delta)=0.$$

The quantity δ is somewhere between the highest and the lowest of α, β, γ. Let us suppose that

$$\alpha-\delta=l,\quad \beta-\delta=m,\quad \delta-\gamma=n,$$

so that l, n are always positive. We get

$$la+mb=nc.$$

But

$$c=100-a-b\,;$$

hence

$$la+mb=n\,(100-a-b),$$

so that

$$(l+n)\,a+(m+n)\,b=100n.$$

Now let us write

$$\xi_2=1,\qquad \eta_2=a\,;$$

$$\xi_3=-1,\quad \eta_3=b\,;$$

leaving ξ_1, η_1 at present undefined. The determinantal equation

$$\begin{vmatrix} \xi_1, & \eta_1, & 1 \\ 1, & a, & 1 \\ -1, & b, & 1 \end{vmatrix} = 0$$

when worked out becomes

$$a(1+\xi_1)+b(1-\xi_1)=2\eta_1.$$

Comparing this with

$$(l+n)a+(m+n)b=100n,$$

we see that we must define ξ_1, η_1 by means of the equations

$$\frac{1+\xi_1}{2\eta_1}=\frac{l+n}{100n}, \quad \frac{1-\xi_1}{2\eta_1}=\frac{m+n}{100n}.$$

It follows that $$\frac{1}{\eta_1}=\frac{l+m+2n}{100n},$$

so that $$\xi_1=\frac{l-m}{l+m+2n}, \quad \eta_1=\frac{100n}{l+m+2n}.$$

In reality the quantities l, m, n are not three quantities, but *two ratios*. It is convenient to put

$$\frac{l}{m}=p, \quad \frac{l+m}{2n}=q.$$

We get $$\xi_1=\frac{p-1}{p+1}\frac{q}{1+q}, \quad \eta_1=\frac{50}{1+q}.$$

It is now clear that for a given value of q, the point (ξ_1, η_1) lies on the straight line

$$\eta=\frac{50}{1+q}.$$

For a given value of p the point (ξ_1, η_1) lies on the locus defined by

$$\xi=\frac{p-1}{p+1}\left(1-\frac{\eta}{50}\right).$$

This is again a straight line passing through the point $\eta=50$. Hence for all values of p we get a family of straight lines passing through the common point $\xi=0$, $\eta=50$. If we put $\eta=0$ in the equation of any such straight line, we get

$$\xi=\frac{p-1}{p+1}.$$

This means that the distance intercepted on the ξ axis between the b and a scales is divided in the ratio $p:1$ or $l:m$.

We now have the nomogram given in Fig. 66, where, for any given values of l, m, n, we get some definite point on the network of straight lines, and the values of a, b given by any straight line passing through this point, and cutting the a, b scales above the ξ axis, satisfy the conditions of the problem. It is clear that for any given values of l, m, n an infinite number of lines pass through the particular point of the network, so that by taking a straight edge through this point, and rotating it round this point as a pivot, from the position in which it passes through the zero graduation on a, in the anti-clockwise sense, till the position in which it passes through the zero graduation on b, we get all possible values of a, b which satisfy the conditions of the problem.

It is worth noticing that if we take the part of the η axis between $\eta=50$ and $\eta=0$, divide it into a hundred equal parts, and graduate it from 0 at $\eta=50$ to 100 at $\eta=0$, we get a scale that can be used as the

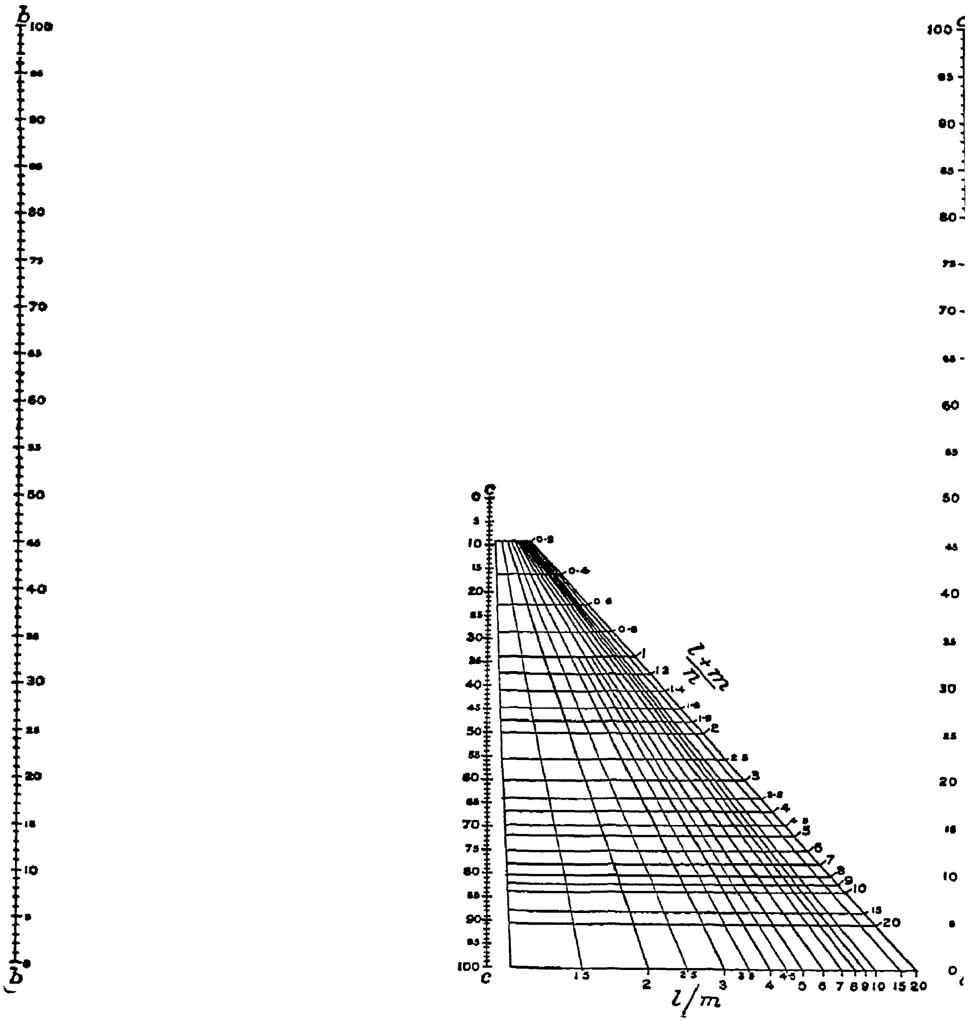


FIG 66.

c scale, so that the values of c can also be read off automatically.

Fig. 66 is drawn for positive value of p, greater than unity. This assumes $\beta > \delta$, or δ between β and γ.

If this is not the case arrange the prices in *ascending* order of magnitude. δ is now between β and γ; $\alpha-\delta$, $\beta-\delta$, $\delta-\gamma$ are all of the same sign, and the nomogram of Fig. 66 can be used.

93. It is immediately clear that we get exactly similar results as in § 91, if the fourth quantity is associated with a in (ξ_2, η_2) or with b in (ξ_3, η_3). In the first case the point (ξ_2, η_2) lies on a network of curves, while (ξ_1, η_1) and (ξ_3, η_3) each lies on a single curve. In the second case the point (ξ_3, η_3) lies on a network of curves, while (ξ_1, η_1) and (ξ_2, η_2) each lies on a single curve.

94. Further Extensions.

It should be fairly obvious to the reader that the method of determinants for four variables can be extended to five or to six quantities connected by appropriate types of equations. If ξ_1, η_1 are functions of x, λ; ξ_2, η_2 are functions of a, μ; while ξ_3, η_3 are functions of b only, then we have by means of the determinantal method a nomogram in which the first point (ξ_1, η_1) lies on a network of curves corresponding to constant λ and constant x respectively; (ξ_2, η_2) lies on a network of curves corresponding to constant μ and constant a respectively; (ξ_3, η_3) lies on a single b curve. The collinearity of three points means that the quantities x, λ, a, μ, b given by these three collinear points all satisfy an equation of the general type

$$X(x, \lambda)\{A'(a, \mu)-B'(b)\}-X'(x, \lambda)\{A(a, \mu)-B(b)\} \\ +\{A(a, \mu)B'(b)-A'(a, \mu)B(b)\}=0,$$

where for ξ_1, η_1 we write $X(x, \lambda)$, $X'(x, \lambda)$, for ξ_2, η_2 we write $A(a, \mu)$, $A'(a_1, \mu)$, for ξ_3, η_3 we write $B(b)$, $B'(b)$.

If now ξ_1, η_1 are functions of x, λ; ξ_2, η_2 of a, μ; and ξ_3, η_3 of b, ν, each of the three points is defined by a network, and the collinearity of three points means that the six quantities $x, \lambda, a, \mu, b, \nu$ corresponding to these three points are connected by an equation of the general type

$$X(x,\lambda)\{A'(a,\mu)-B'(b,\nu)\}-X'(x,\lambda)\{A(a,\mu)-B(b,\nu)\} \\ +\{A(a,\mu)B'(b,\nu)-A'(a,\mu)B(b,\nu)\}=0,$$

where ξ_1, η_1 are written $X(x, \lambda)$, $X'(x, \lambda)$, ξ_2, η_2 are written $A(a, \mu)$, $A'(a, \mu)$, ξ_3, η_3 are written $B(b, \nu)$, $B'(b, \nu)$.

Other applications suggest themselves, but sufficient has been said to indicate the value of the determinantal method and the fruitfulness of its application.

Examples IX.

1. Investigate the nomogram given by the determinantal equation

$$\begin{vmatrix} 0, & x, & 1 \\ 1, & a, & 1 \\ -1, & b, & 1 \end{vmatrix}=0.$$

2. Construct the determinantal equation appropriate to the nomogram for

$$x=2a-\tfrac{1}{2}b.$$

3. Write down the determinantal equation for the nomogram

$$X=\frac{A^2}{B}.$$

4. If ξ_2, η_2 are given functions of a, and ξ_3, η_3 are respectively the same functions of b, show that in the resulting nomogram a and b are measured on the same curve.

Hence construct the nomogram for

$$x=\frac{a-b}{\log a-\log b}$$

by the use of the determinantal form

$$\begin{vmatrix} \frac{1}{x}, & 0, & 1 \\ \frac{\log a}{a}, & \frac{1}{a}, & 1 \\ \frac{\log b}{b}, & \frac{1}{b}, & 1 \end{vmatrix}=a.$$

Show that the equation of the curve for a and b is

$$\xi+\eta\log\eta=0.$$

5. Examine the meaning of the case in which ξ_1, η_1 are given functions of x; ξ_2, η_2 are the same given functions respectively of a; and ξ_3, ξ_2 are the same given functions respectively of b.

6. Put the formula

$$\sin x=a+bx$$

in the determinantal form, using

$$\xi_2=1, \quad \eta_2=a;$$
$$\xi_3=-1, \quad \eta_3=b;$$

with ξ_1, η_1 as functions of x.

7. Construct a nomogram for

$$x^4+\lambda x^3+ax^2+b=0,$$

using

$$\xi_2=-1, \quad \eta_2=a;$$
$$\xi_3=0, \quad \eta_3=b.$$

8. Construct a nomogram for

$$a\sinh x+b\cosh x=c,$$

using

$$\xi_2=0, \quad \eta_2=a;$$
$$\xi_3=1, \quad \eta_3=b$$

Show that the x curves, that is, the curves with constant c, are a family of parabolas, which have a common axis and a common vertex, while the c curves (with constant x) are lines parallel to the a, b scales.

9. Prove by means of the determinantal method for nomograms that there are no real unequal values of three quantities a, b, c, such that

$$a^{b-c}b^{c-a}c^{a-b}=1.$$

10. Prove that there is only one value of x for which the quantity

$$(1+x)^{\frac{1}{x}}$$

has a given positive value, less than e.